ÜBUNGEN ZU NUMERIK ZEITABHÄNGIGER PROBLEME

für den 19.10.2009

---

1. Let $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuous on the set

$$D = \{(t, v) \in \mathbb{R} \times \mathbb{R}^b : t_0 \leq t \leq T,\ \|v - u_0\| \leq b\}.$$

Assume that there are constants A and L such that

$$\|f(t, v)\| \leq A \quad \text{for all } (t, v) \in D$$

and

$$\|f(t, w) - f(t, v)\| \leq L\,\|w - v\| \quad \text{for all } (t, v), (t, w) \in D$$

and assume that $T - t_0 \leq b/A$.

Consider the following initial value problem: Find $u : [t_0, T] \longrightarrow \mathbb{R}$ such that

$$u'(t) = f(t, u(t) \quad t \in (0, T),$$
$$u(t_0) = u_0.$$

Let τ be a subdivision of $[t_0, T]$ with grid points t_j, $j = 0, 1, \ldots, m$ and step sizes $\tau_j = t_{j+1} - t_j$, $j = 0, 1, \ldots, m - 1$.

Show for the corresponding Euler polygon u_τ:

$$\|u_\tau(t) - u_0\| \leq A\,|t - t_0| \quad \text{for all } t \in [t_0, T].$$

2. Since f is uniformly continuous on the compact set D, we have: For each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|t - s| \leq \delta \quad \text{and} \quad \|w - v\| \leq A\,\delta \quad \text{imply} \quad \|f(t, w) - f(s, v)\| \leq \varepsilon$$

for all $(s, v), (t, w) \in D$.

Let $\hat{\tau}$ be a further subdivision of $[t_0, T]$ with grid points $\hat{t}_j$, $j = 0, 1, \ldots, \hat{m}$ such that $\{t_0, t_1, \ldots, t_m\} \subset \{\hat{t}_0, \hat{t}_1, \ldots, \hat{t}_{\hat{m}}\}$ ($\hat{\tau}$ is then called a refinement of τ).

Show:

$$\text{If} \quad |\tau| \leq \delta, \quad \text{then} \quad \|d_k\| \leq \varepsilon\,(t_k - t_{k-1}),$$

where d_k is the difference between $u_{\hat{\tau}}(t_k)$ and $u_{\hat{\tau}}(t_{k-1}) + \tau_{k-1}\, f(t_{k-1}, u_{\hat{\tau}}(t_{k-1}))$.

Hint: Since t_{k-1} is also a grid point of $\hat{\tau}$, the Euler polygon $u_{\hat{\tau}}(t)$ conincides for $t \geq t_{k-1}$ with the Euler polygon that starts at t_{k-1} with the value $u_{\hat{\tau}}(t_{k-1})$. Compare this Euler polygon with $u_{\hat{\tau}}(t_{k-1}) + (t - t_{k-1})\, f(t_{k-1}, u_{\hat{\tau}}(t_{k-1}))$ for $t \in [t_k - t_{k-1}]$.

3. Consider, for the subdivision τ, the Euler polygon that starts at t_k with the value $u_{\hat{\tau}}(t_k)$ and the Euler polygon that starts at t_k with the value $u_{\hat{\tau}}(t_{k-1}) + \tau_{k-1}\, f(t_{k-1}, u_{\hat{\tau}}(t_{k-1}))$. The difference between these two Euler polygons at some point $t \geq t_k$ is denoted by d'_k.

   Show
$$\|d'_k\| \leq e^{L\,(t-t_k)}\|d_k\|.$$

   Hint: Show for two sequences (v_j) and (w_j), given by
$$v_{j+1} = v_j + \tau_j\, f(t_j, v_j), \quad w_{j+1} = w_j + \tau_j\, f(t_j, w_j),$$
   that
$$\|w_{j+1} - v_{j+1}\| \leq (1 + \tau_j\, L)\, \|w_j - v_j\| \leq e^{\tau_j\, L}\, \|w_j - v_j\|.$$

4. Show for $e(t) = u_{\hat{\tau}}(t) - u_{\tau}(t)$ with $t_j < t \leq t_{j+1}$ the following estimate:
$$\begin{aligned}\|e\| &\leq \varepsilon\, \left[e^{L\,(t-t_1)}(t_1 - t_0) + e^{L\,(t-t_2)}(t_2 - t_1) + \ldots + e^{L\,(t-t_j)}(t_j - t_{j-1}) + (t - t_j)\right] \\ &\leq \varepsilon \int_{t_0}^{t} e^{L\,(t-s)}\, ds = \frac{\varepsilon}{L}\left(e^{L\,(t-t_0)} - 1\right).\end{aligned}$$

5. Show that u_τ converges uniformly to a continuous function u, if $|\tau| \to 0$.

   Hint: Verify the Cauchy criterion: For each $\varepsilon > 0$ there is a $\delta > 0$ such that
$$|\tau|, |\hat{\tau}| \leq \delta \quad \text{implies} \quad \max_{t\in[t_0,T]} \|u_{\hat{\tau}}(t) - u_\tau(t)\| \leq \varepsilon$$
   for all subdivisions τ and $\hat{\tau}$.

   Hint: Use the triangle inequality with a third subdvision $\hat{\hat{\tau}}$, which is a refinement of both subdivisions τ and $\hat{\tau}$.

6. Show that the limit u of the Euler polygons u_τ for $|\tau| \to 0$ is differentiable with derivative $f(t, u(t))$ at t.

   Hint: Let t be a grid point of the subdivision τ. Show (see Exercise 2):
$$\|u_\tau(t + \delta) - u_\tau(t) - \delta\, f(t, u_\tau(t))\| \leq \varepsilon(\delta)\, \delta$$
   with
$$\varepsilon(\delta) = \sup\left\{\|f(t, w) - f(s, v)\| : |t - s| \leq \delta,\ \|w - v\| \leq A\,\delta,\ (t, w), (s, v) \in D\right\}.$$
   Take the limit $\delta \to 0$.