Chapter 2

Euler's Method

See also: Hairer, Nørsett, Wanner, [8], I.7.

Leonhard Euler: 1707 - 1788, Swiss mathematician

Initial value problem (IVP):

$$\begin{aligned} u'(t) &= f(t, u(t)), \quad t \in (t_0, T) \\ u(t_0) &= u_0 \end{aligned}$$

subdivision of $[t_0, T]$:

$$t_0 < t_1 < \ldots < t_m = T$$

Notations:

- $\tau_j = t_{j+1} - t_j$: step size ($t_{j+1} = t_j + \tau_j$)
- $\tau = (\tau_0, \tau_1, \ldots, \tau_{m-1})$.
- $|\tau| = \max_{j=0,1\ldots,m-1} \tau_j$

For $t \in [t_0, t_1]$:

$$u(t) \approx u(t_0) + (t - t_0)\, u'(t_0) = u_0 + (t - t_0)\, f(t_0, u_0) = u_\tau(t)$$

$$u_1 = u_\tau(t_1) = u_0 + \tau_0\, f(t_0, u_0).$$

For $t \in [t_1, t_2]$:

$$u(t) \approx u(t_1) + (t - t_1)\, u'(t_1) \approx u_1 + (t - t_1)\, f(t_1, u_1) = u_\tau(t)$$

$$u_2 = u_\tau(t_2) = u_1 + \tau_1\, f(t_1, u_1).$$

and so on.
$u_\tau(t)$ is a polygonal approximation of $u(t)$ (Euler polygon). It connects the points (t_j, u_j) with

$$u_{j+1} = u_j + \tau_j\, f(t_j, u_j), \quad j = 0, 1, \ldots, m - 1.$$

Euler's method (1768).

Theorem 2.1 (Cauchy, 1789-1857, French mathematician)**.** *Let f be continuous on D, $\|f\|$ bounded by A on D, and let f satisfy the Lipschitz condition*

$$\|f(t,w) - f(t,v)\| \le L\,\|w - v\|$$

on D, with

$$D = \{(t,v) \in \mathbb{R} \times \mathbb{R}^n : t_0 \le t \le T, \|v - u_0\| \le b\}.$$

If $T - t_0 \le b/A$, then we have:

a) For $|\tau| \to 0$, the Euler polygons converge uniformly to a continuous function $u(t)$.

b) $u(t)$ is continuously differentiable and solves (IVP) on $[t_0, T]$.

c) There is no other solution of (IVP) on $[t_0, T]$.

Proof. Let τ be a subdivision of $[t_0, T]$ with grid points t_j and $\hat{\tau}$ a refinement of τ. Let $t \in [t_0, T]$ with $t_j < t \le t_{j+1}$.

- Then (see tutorial):
$$\|u_\tau(t) - u_0\| \le A\,|t - t_0|.$$
Therefore, $(t, u_\tau(t)) \in D$.

- f is uniformly continuous on the compact set D:
Therefore, for each $\varepsilon > 0$, there is a $\delta > 0$ such that
$$|t - s| \le \delta \quad \text{and} \quad \|w - v\| \le A\,\delta \quad \text{imply} \quad \|f(t,w) - f(t,v)\| \le \varepsilon$$
for all $(s,v), (t,w) \in D$.
Then (see tutorial):
$$\|d_k\| \le \varepsilon\,(t_k - t_{k-1}) \quad \text{if} \quad |\tau| \le \delta$$
with
$$d_k = u_{\hat{\tau}}(t_k) - \left[u_{\hat{\tau}}(t_{k-1}) + \tau_{k-1}\, f(t_{k-1}, u_{\hat{\tau}}(t_{k-1}))\right].$$

- Let d_k' be the difference of the values of Euler's method at t using subdivision τ starting in t_k with initial values $u_{\hat{\tau}}(t_k)$ and $u_{\hat{\tau}}(t_{k-1}) + \tau_{k-1}\, f(t_{k-1}, u_{\hat{\tau}}(t_{k-1}))$, respectively.
Then (see tutorial):
$$\|d_k'\| \le e^{L\,(t - t_k)}\,\|d_k\|.$$

- Then, for $e = u_{\hat{\tau}}(t) - u_\tau(t)$, (see tutorial):
$$\begin{aligned}\|e\| &\le \varepsilon\,\left[e^{L\,(t-t_1)}(t_1 - t_0) + e^{L\,(t-t_2)}(t_2 - t_1) + \ldots + e^{L\,(t-t_j)}(t_j - t_{j-1}) + (t - t_j)\right] \\ &\le \varepsilon \int_{t_0}^{t} e^{L\,(t-s)}\,ds = \frac{\varepsilon}{L}\left(e^{L\,(t-t_0)} - 1\right).\end{aligned}$$

- Therefore, u_τ converges uniformly to a continuous function u, if $|\tau| \to 0$, see tutorial.
- u solves (IVP) (see tutorial).
- u is the only solution to (IVP) (see tutorial).

□

Error estimate:

- Global error:
  If $|\tau| \leq \delta$, then
  $$\|u(t) - u_\tau(t)\| \leq \frac{\varepsilon}{L}\left(e^{L\,(t-t_0)} - 1.\right)$$
  Euler's method is convergent.

  If, additionally,
  $$\|f(t,v) - f(s,v)\| \leq M\,|t-s| \quad \text{on } D, \tag{2.1}$$
  then, for $|t-s| \leq \delta$ and $\|w-v\| \leq A\,\delta$,
  $$\begin{aligned}\|f(t,w) - f(s,v)\| &\leq \|f(t,w) - f(t,v)\| + \|f(t,v) - f(s,v)\| \\ &\leq L\,\|w-v\| + M\,|t-s| \leq (AL+M)\,\delta\end{aligned}$$
  and, therefore,
  $$\|u(t) - u_\tau(t)\| \leq \frac{AL+M}{L}\left(e^{L\,(t-t_0)} - 1\right)\,|\tau|^1$$
  Euler's method is of convergence order 1.

  Sufficient conditions for the Lipschitz conditions:
  $$\left\|\frac{\partial f}{\partial u}(t,v)\right\| \leq L, \quad \left\|\frac{\partial f}{\partial t}(t,v)\right\| \leq M \quad \text{on } D.$$

- Local error:
  $$d_\tau(t_{j+1} = u(t_{j+1}) - \left[u(t_j) + \tau_j\,f(t_j, u(t_j))\right]$$
  We have:
  $$\|d_\tau(t_{j+1})\| \leq \varepsilon\,\tau_j,$$
  Euler's method is consistent with (IVP).
  Under the additional condition (2.1), we have
  $$\|d_\tau(t_{j+1})\| \leq (AL+M)\tau_j^2.$$
  Euler's method is of consistency order 1.

- Approximation error

  Euler's method as a finite difference method: $u'(t_j)$ is replaced by the forward difference quotient:

$$\frac{1}{\tau_j}\big(u(t_{j+1}) - u(t_j)\big)$$

  This leads to the FDM

$$\frac{1}{\tau_j}\big(u_{j+1} - u_j\big) = f(t_j, u_j)$$

  for the approximation u_j of $u(t_j)$. The corresponding approximation error is given by

$$\psi_\tau(t_{j+1}) = \frac{1}{\tau_j}\big(u(t_{j+1}) - u(t_j)\big) + f(t_j, u(t_j)).$$

  Observe

$$\psi_\tau(t_{j+1}) = \frac{1}{\tau_j}\, d_\tau(t_{j+1}).$$

  We have the following estimates:

$$\|\psi_\tau(t_{j+1})\| \leq \varepsilon$$

  Euler's method is consistent with (IVP).

$$\|\psi_\tau(t_{j+1})\| \leq (AL + M)\tau_j^1$$

  Euler's method is of consistency order 1.

Local variant:

Theorem 2.2. *Let $U \subset \mathbb{R}^n \times \mathbb{R}$ be an open set and let f and $\frac{\partial f}{\partial u}$ continuous on U. Then, for every $(t_0, u_0) \in U$, there exists a unique solution of (IVP), which can be continued up to the boundary of U.*

Example. *Restricted three body problem*

$$y_1'' = y_1 + 2y_2 - \mu' \,\frac{y_1 + \mu}{D_1^3} - \mu\,\frac{y_1 - \mu'}{D_2^3}$$
$$y_2'' = y_2 - 2y_1 - \mu' \,\frac{y_2}{D_1^3} - \mu\,\frac{y_2}{D_2^3}$$

with

$$D_1 = \big[(y_1 + \mu)^2 + y_2^2\big]^{1/2}, \quad D_2 = \big[(y_1 - \mu')^2 + y_2^2\big]^{1/2}$$

and $\mu = 0.012277471$, $\mu' = 1 - \mu$. Initial conditions: $y_1(0)$, $y_1'(0)$, $y_2(0)$, $y_2'(0)$ are given.