55 Consider the sequence $u_\tau \in X_\tau$ according to the explicit Euler method and the perturbed sequence $v_\tau \in Y_\tau$ defined by

$$\begin{aligned} u_{j+1} &= u_j + \tau_j\, f(t_j,\, u_j)\,, \\ v_{j+1} &= v_j + \tau_j \left[f(t_j,\, v_j) + y_{j+1} \right], \end{aligned}$$

with $v_0 = u_0 + y_0$ and the perturbations $y_\tau \in Y_\tau$ (see your Lecture notes). Assume further, that f satisfies the Lipschitz condition

$$\|f(t,\, v) - f(t,\, w)\| \;\leq\; L\, \|v - w\| \qquad \forall t, v, w\,.$$

Show that

$$\|v_j - u_j\| \;\leq\; e^{(t_j - t_0)L} \|y_0\| + \frac{1}{L}\left(e^{(t_j - t_0)L} - 1 \right) \max_{k=1,\ldots,j} \|y_k\|\,.$$

Hint: Show and use

$$e^{(t_j - t_k)L}\, \tau_{k-1} \;\leq\; \int_{t_{k-1}}^{t_k} e^{(t_j - t)L}\, dt\,.$$

56 The definitions of consistency, stability, and convergence depend on the underlying norms. In the following we use $\|\cdot\|_{X_\tau}$ instead of $\|\cdot\|_{Y_\tau}$. Using Exercise **55**, derive the estimate

$$\|e_\tau\|_{X_\tau} \;\leq\; C\, \|\psi_\tau\|_{X_\tau}\,.$$

Hint: follow your lecture notes.
For exact solutions $u \in C^2([0,\, T],\, \mathbb{R}^n)$ show an estimate of the form

$$\|\psi_\tau\|_{X_\tau} \;\leq\; K\, \tau$$

(*Hint:* $K = \max_{s \in [0,\, T]} |u''(s)| < \infty$), and conclude a corresponding estimate for the global error.

57 Consider the 2-stage Runge-Kutta method

$$\begin{aligned} g_1 &= u_j\,, \\ g_2 &= u_j + \tau_j\, a_{21}\, f(t_j,\, g_1)\,, \\ u_{j+1} &= u_j + \tau_j \big[b_1\, f(t_j,\, g_1) + b_2\, f(t_j + c_2\, t_j,\, g_2) \big]\,, \end{aligned}$$

for the solution of the initial value problem

$$\begin{aligned} u'(t) &= f(t,\, u(t)) \qquad \forall t \in (0,\, T)\,, \\ u(0) &= u_0\,, \end{aligned}$$

with $f : [0,\, T] \times \mathbb{R} \to \mathbb{R}$ sufficiently smooth. Provide a Taylor series expansion of the local error $d(t + \tau)$ of the form

$$d(t + \tau) \;= A_0 + \tau\, A_1 + \tau^2\, A_2 + \tau^3\, A_3 + \mathcal{O}(\tau^4)$$

with the expressions A_i only depending on a_{21}, b_1, b_2, c_2, f and its derivatices, but not on τ.
Hints: Similar but more general than the analysis of the explicit midpoint rule in your lecture notes.

58 Continue exercise 57 and find necessary and sufficient conditions on the coefficient a_{21}, b_1, b_2, c_2 such that the consitency order is at least 2, i.e., that for all sufficiently smooth functions f, we have

$$A_0 = A_1 = A_2 = 0\,.$$

59 Show that

$$R(z) \;=\; 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$$

is the stability function of the classical Runge-Kutta method of order 4,

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

and show that

$$e^z - R(z) \;=\; \mathcal{O}(z^5) \qquad \text{as } z \to 0\,.$$

60 Let M_h be the mass matrix of the one-dimensional model problem

$$\begin{aligned} -u''(x) &= f(x)\,, \qquad x \in (0,\,1) \\ u(0) &= 0\,, \\ u'(1) &= g_1\,, \end{aligned}$$

(the time t is omitted here), semi-discretized using a spacial grid. Let $\varphi_1, \ldots, \varphi_{N_h}$ denote the basis functions (φ_0 is excluded due to the Dirichlet boundary condition). Let $\overline{M}$ be the matrix defined by

$$\overline{M}_{ik} = (\varphi_i,\,\varphi_j)_h\,, \quad \text{where} \quad (v,\,w)_h := \sum_{k=1}^{N_h} \frac{h_k}{2}\big[v(x_{k-1}\,w(x_{k-1}) + v(x_k)\,w(x_k)\big]\,.$$

Show that

$$\overline{M}_{ii} \;=\; \sum_{k=1}^{N_h} M_{ik} \qquad \forall i \geq 2\,.$$