Consider the initial-value problem

$$\begin{aligned} u'(t) &= f(t,\, u(t)) \qquad \text{for } t > 0\,, \\ u(0) &= u_0\,, \end{aligned} \tag{9.1}$$

where $u : \mathbb{R}_0^+ \to X$ and $f : \mathbb{R}_0^+ \times X \to X$ with the Banach space $(X,\, \|\cdot\|)$.

49 Assume that there exists a constant $L > 0$ such that

$$\|f(t,\, w) - f(t,\, v)\| \;\le\; L\|w - v\| \qquad \forall t \in \mathbb{R}_0^+ \quad \forall v,\, w \in X\,.$$

Show that, for each t_j and u_j, there exists a unique solution u_{j+1} of the equation

$$u_{j+1} \;=\; u_j + \tau\, f(t_j + \tau,\, u_{j+1})\,,$$

if $\tau < 1/L$.

Hint: Apply Banach's fixed point theorem.

50 Assuming X is a Hilbert space with the inner product $(\cdot,\, \cdot)$ and that

$$\|f(t,\, w) - f(t,\, v)\| \;\le\; L\,\|w - v\| \qquad \forall t \in \mathbb{R}_0^+ \quad \forall v,\, w \in X\,,$$

and

$$\big(f(t,\, w) - f(t,\, v),\, w - v\big) \;\le\; 0 \qquad \forall t \in \mathbb{R}_0^+ \quad \forall v,\, w \in X\,,$$

show that, for each $\tau > 0$, t_j and u_j there exists a unique solution u_{j+1} of the equation

$$u_{j+1} \;=\; u_j + \tau\, f(t_j + \tau,\, u_{j+1})\,.$$

Hint: Apply Banach's fixed point theorem to the following equivalent equation

$$u_{j+1} = G(u_{j+1}) := (1 - \rho)u_{j+1} + \rho\big[u_j + \tau\, f(t_j + \tau,\, u_{j+1})\big]\,,$$

for some parameter $\rho \in (0,\, 1)$. Estimate

$$\|G(w) - G(v)\|^2 \;=\; \big(G(w) - G(v),\, G(w) - G(v)\big)\,,$$

and choose $\rho \in (0,\, 1)$ such that G is a contraction.

For the following exercises we consider the implicit Euler method, i. e.,

$$u_{j+1} \;=\; u_j + \tau\, f(t_{j+1},\, u_{j+1})\,.$$

Let

$$\psi_\tau(t + \tau) \;=\; \frac{1}{\tau}\big[u(t + \tau) - u(t)\big] - f(t + \tau,\, u(t + \tau))$$

denote the consistency error of the implicit Euler method, where $u(t)$ is the solution of (9.1). Furthermore,

$$e_k := u(t_k) - u_k$$

denotes the global error.

51 Show that the following estimate holds if $u''(\cdot)$ exists:

$$\|\psi_\tau(t+\tau)\| \;\le\; \int_t^{t+\tau} \|u''(s)\|\, ds\,.$$

52 Show that

$$u(t_{j+1}) \;=\; u(t_j) + \tau\, f(t_{j+1},\, u(t_{j+1})) + \tau\, \psi_\tau(t_{j+1})\,,$$

and

$$e_{j+1} \;=\; e_j + \tau\big[f(t_{j+1},\, u(t_{j+1})) - f(t_{j+1},\, u_{j+1})\big] + \tau\, \psi_\tau(t_{j+1})\,. \tag{9.2}$$

53 Let the assumptions of exercise **50** be fulfilled. Show that the following estimate holds:

$$\|e_{j+1}\| \;\le\; \|e_j\| + \tau\|\psi_\tau(t_{j+1})\|\,.$$

Hint: Multiply (9.2) by e_{j+1} and apply Cauchy's inequality to the right hand side.

54 Let the assumptions of exercise **50** be fulfilled. Show that

$$\|u(t_j) - u_j\| \;\le\; \tau \int_0^{t_j} \|u''(s)\|\, ds\,,$$

if $u''(\cdot)$ exists.