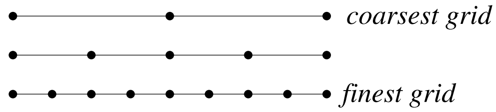
In this tutorial we consider the MDS (multilevel diagonal scaling) preconditioner. Let $\{\mathcal{T}_\ell\}_{1\le\ell\le L}$ be a family of subdivisions of the interval $\Omega = (0,\,1)$ given by the nodes

$$0 \le i \le N_\ell\,.$$

In the following we fix a uniform mesh $\mathcal{T}_1$ and a fixed number of levels $L \ge 1$, and we define $\mathcal{T}_2, \ldots, \mathcal{T}_L$ recursively by refinement as shown below.



The mesh $\mathcal{T}_1$ is called *coarsest grid*, whereas $\mathcal{T}_L$ is the *finest grid*. When comparing two subsequent grids $\mathcal{T}_\ell$, $\mathcal{T}_{\ell+1}$, these are called *coarse* and *fine grid*, respectively.

37 Let w_ℓ be a finite element function on the coarse grid $\mathcal{T}_\ell$:

$$w_\ell(x) = \sum_{i=0}^{N_\ell} w_{\ell,i}\,\varphi_{\ell,i}(x)\,.$$

Find a representation of w_ℓ using the basis functions $\{\varphi_{\ell+1,i}\}_i$ associated to the fine grid $\mathcal{T}_{\ell+1}$ such that

$$w_\ell(x) = \sum_{i=0}^{N_{\ell+1}} w_{\ell+1,i}\,\varphi_{\ell+1,i}(x)\,.$$

Represent the relation between the coefficient vectors $\underline{w}_{\ell+1} = (w_{\ell+1,i})_{i=0,\ldots,N_{\ell+1}}$ and $\underline{w}_\ell = (w_{\ell,i})_{i=0,\ldots,N_\ell}$ by an $N_{\ell+1} \times N_\ell$ matrix $I_\ell^{\ell+1}$ such that

$$\underline{w}_{\ell+1} = I_\ell^{\ell+1}\,\underline{w}_\ell\,.$$

38 Let $R : H^1(0,\,1) \to \mathbb{R}$ be a continuous linear functional. Furthermore, for the coefficient vector $\underline{r}_{\ell+1} = (r_{\ell+1,i})_{i=0,\ldots,N_{\ell+1}}$ defined by the relation

$$r_{\ell+1,i} := \langle R,\, \varphi_{\ell+1,i}\rangle\,,$$

we have

$$\langle R,\, v_{\ell+1}\rangle = \sum_{i=0}^{N_{\ell+1}} r_{\ell+1,\,i}\,v_{\ell+1,\,i} = (\underline{r}_{\ell+1},\, \underline{v}_{\ell+1})_{\ell_2}\,.$$

Find a representation of the evaluation of this functional for a finite element function v_ℓ defined on the coarse grid $\mathcal{T}_\ell$:

$$\langle R,\, v_\ell\rangle = \sum_{i=0}^{N_\ell} r_{\ell,i}\,v_{\ell,i} = (\underline{r}_\ell,\, \underline{v}_\ell)_{\ell_2}\,.$$

Show the following relation between the coefficient vectors $\underline{r}_{\ell+1} = (r_{\ell+1,i})_{i=0,\ldots,N_{\ell+1}}$ and $r_\ell = (r_{\ell,i})_{i=0,\ldots,N_\ell}$:

$$\underline{r}_\ell = I_{\ell+1}^{l}\,\underline{r}_{\ell+1}\,, \qquad \text{with } I_{\ell+1}^{l} = (I_\ell^{\ell+1})^\top\,.$$

Hint:

$$(\underline{r}_\ell,\, v_\ell)_{\ell_2} = \langle R,\, v_\ell\rangle = (\underline{r}_{\ell+1},\, \underline{v}_{\ell+1})_{\ell_2} = (\underline{r}_{\ell+1},\, I_\ell^{\ell+1}\underline{v}_\ell)_{\ell_2}\,.$$

39 Write a function `RefineUniform(↓coarsemesh, ↑finemesh)`, which computes the refined grid $\mathcal{T}_{\ell+1}$ (`finemesh`) starting from the coarse mesh $\mathcal{T}_\ell$ (`coarsemesh`) as described above.

40 Write a function `Prolongate(↓coarsevector, ↑finevector)` which computes $\underline{w}_{\ell+1} = I_\ell^{\ell+1}\, w_\ell$, where `coarsevector`$=\underline{w}_\ell$ and `finevector`$=\underline{w}_{\ell+1}$.

Write a function `Restrict(↓finevector, ↑coarsevector)` which computes $\underline{w}_\ell = I_{\ell+1}^{l}\, \underline{w}_{\ell+1}$, where `finevector`$=\underline{w}_{\ell+1}$ and `coarsevector`$=\underline{w}_\ell$.

41–42 Implement the MDS preconditioner C_{MDS}^{-1} for a hierarchy of recursively refined grids $\mathcal{T}_1, \ldots, \mathcal{T}_\ell$, i. e., implement the operation

$$\underline{w}_\ell = C_{\text{MDS}}^{-1}\, \underline{r}_\ell \,.$$

For $\ell = 1, \ldots, L$ we define $D_\ell = \operatorname{diag}(K_\ell)$.

1. If there is only one grid ($L = 1$), then the MDS preconditioner coincides with the Jacobi preconditioner, i. e.,

$$\underline{w}_1 = D_1^{-1}\, \underline{r}_1 \,.$$

2. For two grids $L = 2$, the correction $\underline{w}_2$ is defined as the sum of
   - the correction obtained by the Jacobi preconditioner applied to the residual $\underline{r}_2$ on the fine grid,
   - and the (prolongated) correction obtained by the Jacobi preconditioner on the coarse grid for the (restricted) residual $\underline{r}_1$, i. e.,

$$\underline{w}_2 = D_2^{-1}\, \underline{r}_2 + I_1^2\, \underline{w}_1 \,,$$

   with

$$\underline{w}_1 = D_1^{-1}\, \underline{r}_1, \qquad \text{where } \underline{r}_1 = I_2^1\, \underline{r}_2 \,.$$

3. For a hierarchy of grids $\mathcal{T}_1, \ldots, \mathcal{T}_\ell$ we have the recursive definition:

$$\begin{aligned} C_{\text{MDS},1}^{-1} &= D_1^{-1}\,, \\ C_{\text{MDS},l}^{-1} &= D_\ell^{-1} + I_{\ell-1}^{l}\, C_{\text{MDS},l-1}\, I_\ell^{\ell-1}\,. \end{aligned}$$

   and $C_{\text{MDS}} := C_{\text{MDS},L}$.

Hint: The operation $\underline{w}_\ell = C_{\text{MDS},l}^{-1}\, \underline{r}_\ell$ for $l > 1$ is equivalent to

$$\begin{aligned} \underline{r}_{\ell-1} &= I_\ell^{\ell-1}\, \underline{r}_\ell\,, \\ \underline{w}_{\ell-1} &= C_{\text{MDS},l-1}^{-1}\, \underline{r}_{\ell-1}\,, \\ \underline{w}_\ell &= I_{\ell-1}^{l}\, \underline{w}_{\ell-1}\,. \end{aligned}$$

Hence, use a recursive function like

```
void MDS (int l, const Vector& r, Vector& w)
{
  . . .
  if (l == 1)
    {
      w = JacobiPreconditioner.solve (1, r);
    }
```

```
  else
    {
      w = JacobiPreconditioner.solve (l, r);
      Restrict (r, r_coarse);
      MDS (l-1, r_coarse, w_coarse);
      Prolongate (w_coarse, w_fine);
      w += w_fine;
    }
}
```

Preview to Tutorial 08 (in January):

- Embed your MDS code in a class `MDSPreconditioner` where you have a member function `solve` similar to the one in your existing Jacobi preconditioner class, e. g., `Vector MDSPreconditioner::solve (const Vector& r)`.

  Consider a FEM discretized boundary value problem of your choice and solve it using the preconditioned CG method using the MDS preconditioner.

- Numerical comparisons of CG with Jacobi and MDS (with different levels etc.).