19 Let $\mathcal{T}_h$ be an equidistant subdivision of (0, 1). Show that there exists a constant $C_1 > 0$ such that

$$|v - I_h v|_{H^1(0,1)} \leq C_1 \, h \, \|v''\|_{L_2(0,1)} \qquad \forall v \in C^2(0, 1)\,. \tag{4.1}$$

Hint: Perform the analogous steps as for the L_2-estimate in lecture notes.

20 Closure principle: Show that all expressions in (4.1) are continuous with respect to the H^2-norm. (Then, it follows that the inequality holds for all $v \in H^2(0, 1)$, because one can show that $C^2(0, 1)$ is dense in $H^2(0, 1)$.)
Hint: Show Lipschitz conditions for the left and the right hand side of (4.1).

21 Let $\mathcal{T}_h$ be an equidistant subdivision of (0, 1) and let V_{0h} be the space of piecewise affine linear functions vanising at 0. Estimate the condition number of the stiffness matrix K_h of our model problem from below and show that there exists a constant $C_2 > 0$ such that i. e., show an estimate of the form

$$\kappa(K_h) \;\geq\; C_2 \, h^{-2}\,.$$

Hint: Use the Rayleigh quotient for the special vector $\underline{v}_h = (1, 0, \ldots, 0)^\top$ in order to obtain a loer bound for $\lambda_{\max}(K_h)$. For an upper bound of $\lambda_{\min}(K_h)$ use the Rayleigh quotient for the special vector $\underline{v}_h = (h, 2h, 3h, \ldots, 1)^\top$.

22 Let $\mathcal{T}_h$ be an equidistant subdivision of (0, 1) and let V_{0h} be the space of piecewise affine linear functions vanising at 0. Estimate the condition number of the mass matrix M_h from above, i. e., show an estimate of the form

$$\kappa(M_h) \;\leq\; C_3 \, h^{\beta}\,,$$

for some constants $C_3 > 0$ and $\beta \in \mathbb{R}$.

23 Let Ω be a bounded domain in $\mathbb{R}^2$, and let $\Gamma_D,\ \Gamma_N \subset \partial\Omega$ be disjoint such that $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial\Omega$. Derive the variational formulation for the following boundary value problem:

$$\begin{aligned} -\Delta u(x) &= f(x) & \qquad \forall x \in \Omega\,, \\ u(x) &= g_D(x) & \qquad \forall x \in \Gamma_D\,, \\ \frac{\partial u}{\partial n}(x) &= g_N(x) & \qquad \forall x \in \Gamma_N\,. \end{aligned}$$

In particular specify V, V_0 and V_g.

24 Let $\widehat{T} := \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x + y \leq 1\}$ be the two-dimensional reference element with the corner points $\xi_0 = (0, 0)$, $\xi_1 = (1, 0)$, and $\xi_2 = (0, 1)$ Let $\widehat{\varphi}_0$, $\widehat{\varphi}_1$, and $\widehat{\varphi}_2$ denote the affine linear functions on $\widehat{T}$ which fulfill

$$\widehat{\varphi}_i(\xi_j) = \delta_{ij} \qquad \forall i,\ j \in \{0,\ 1,\ 2\}\,.$$

Derive an explicit formula for $\widehat{\varphi}_0$, $\widehat{\varphi}_1$, and $\widehat{\varphi}_2$ in terms of $\xi = (\xi^{(1)}, \xi^{(2)})$.