In the lecture, the coercivity of the bilinear form

$$a(w,\,v) \;=\; \int_0^1 \big[a(x)\,w'(x)\,v'(x) + b(x)\,w'(x)\,v'(x) + c(x)\,w(x)\,v(x)\big]\,dx\,, \qquad (2.1)$$

on the space $V_0 = \{v \in H^1(0,\,1) : v(0) = 0\}$ has been shown for the special case $a(x) = 1$, $b(x) = 0$, $c(x) = 0$. In the following three exercises, we consider more general cases with $a,\,b,\,c \in L_\infty(0,\,1)$. Throughout, you will have to use the estimate

$$a(v,\,v) \;\geq\; a_0\,|v|^2_{H^1(0,\,1)} + \int_0^1 b(x)\,v'(x)\,v(x)\,dx + c_0\,\|v\|^2_{L_2(0,\,1)} \qquad (2.2)$$

(which is rather easily shown), where $a_0 = \inf\limits_{x\in(0,\,1)} a(x)$ and $c_0 = \inf\limits_{x\in(0,\,1)} c(x)$.

07 Show the coercivity of $a(w,\,v)$ on $V_0 = \{v \in H^1(0,\,1) : v(0) = 0\}$ under the assumptions

$$a_0 > 0\,, \qquad C_F\,\|b\|_{L_\infty(0,\,1)} < a_0\,, \qquad c_0 \geq 0\,,$$

where C_F is the constant in Friedrichs' inequality.
Hint: Use Cauchy's inequality to show the estimate

$$\int_0^1 b(x)\,v'(x)\,v(x)\,dx \;\geq\; -\|b\|_{L_\infty(0,\,1)}\,|v|_{H^1(0,\,1)}\,\|v\|_{L_2(0,\,1)}$$

and use it to bound the second term on the right hand side of (2.2).

08 Show the coerivity of $a(w,\,v)$ on the whole space $H^1(0,\,1)$ under the assumptions

$$a_0 > 0\,, \qquad \|b\|_{L_\infty(0,\,1)} < 2\sqrt{a_0\,c_0}\,, \qquad c_0 > 0\,.$$

Hint: Using the estimates above you should be able to obtain

$$a(v,\,v) \;\geq\; q\big(\|v\|_{L_2(0,\,1)},\,|v|_{H^1(0,\,1)}\big)\,,$$

with $q(\xi_0,\,\xi_1) = a_0\,\xi_1^2 - \|b\|_{L_\infty(0,\,1)}\,\xi_1\,\xi_0 + c_0\,\xi_0^2$. Show and use that $q(\xi_0,\,\xi_1) \geq a_0\,C\,\xi_1^2$ and $q(\xi_0,\,\xi_1) \geq c_0\,C\,\xi_0^2$ with $C = 1 - \|b\|^2_{L_\infty(0,\,1)}/(4\,a_0\,c_0)$.

The last two exercises show that we have coercivity if $b(x)$ is in a certain sense small compared to $a(x)$ or $c(x)$. The following exercise shall show that coercivity is also possible under certain assumptions if $b(x)$ is large.

09 Show the coercivity of $a(w,\,v)$ on the space $V_0 = \{v \in H^1(0,\,1) : v(0) = 0$ under the assumptions

$$a_0 > 0\,, \qquad b(x) = b \geq 0\,, \qquad c_0 \geq 0\,,$$

where b is a constant.
Hint: Show and use that

$$b\int_0^1 v'(x)\,v(x)\,dx \;=\; \frac{b}{2}\,v(x)^2\Big|_0^1 \;\geq\; 0 \qquad \forall v \in V_0\,.$$

10 Show *Poincaré's inequality*: There exists a constant $C_P > 0$ such that

$$\|v\|_{L_2(0,\,1)} \;\leq\; C_P\Big\{\Big(\int_0^1 v(x)\,dx\Big)^2 + |v|^2_{H^1(0,\,1)}\Big\}^{1/2} \qquad \forall v \in H^1(0,\,1)\,.$$

Hint: Integrate the identity

$$v(y) \;=\; v(x) + \int_x^y v'(z)\,dz$$

over the whole interval $(0,\,1)$ with respect to x. The rest of the proof is then similar to the one of Friedrichs' inequality (see your lecture notes).

11 Take a look at exercise **04** on the pure Neumann problem and show that the weak formulation (1.3) has a solution if and only if (1.4) holds, and that the solution is unique up to an additive constant.
Hint: Use Poincaré's inequality to show the coercivity of $a(w,\,v)$ on $\widehat{V}$.

12 Let V be a Hilbert space, $a(\cdot,\,\cdot) : V \times V$ a symmetric bilinear form satisfying $a(v,\,v) \geq 0$ for all $v \in V$, and $f \in V^*$ with $V_0 \subset V$. Show directly that the variational formulation

$$\text{find } u \in V_g : \quad a(u,\,v) = \langle f,\,v\rangle \quad \forall v \in V_0$$

with $V_g = g + V_0$ is equivalent to the minimization problem

$$J(u) = \inf_{v \in V_g} J(v) \qquad \text{with} \qquad J(v) = \frac{1}{2}a(v,\,v) - \langle f,\,v\rangle\,.$$

Hint: Modify the corresponding proof from your lecture notes, where the special case $a(u,\,v) = (u,\,v)_V$ with $V_0 = V_g = V$ is treated.