01 Show that we can write each linear second order partial differential equation

$$-\big(a(x)\,u'(x)\big)' + b(x)\,u'(x) + c(x)\,u(x) = f(x)\,, \tag{1.1}$$

with $a \in C^1(0,\,1)$ and $b,\,c \in C(0,\,1)$, also in the form

$$\bar{a}(x)\,u''(x) + \bar{b}(x)\,u'(x) + c(x)\,u(x) = f(x)\,, \tag{1.2}$$

for suitable functions $\bar{a} \in C^1(0,\,1)$ and $\bar{b} \in C(0,\,1)$. Show also the reverse direction.

02 Derive the variational formulation for the following two boundary value problems:

$$\text{(a)} \qquad \begin{cases} -u''(x) + u(x) &= f(x) \qquad \text{for } x \in (0,\,1) \\ u(0) &= g_0 \\ u(1) &= g_1 \end{cases}$$

$$\text{(b)} \qquad \begin{cases} -u''(x) + u(x) &= f(x) \qquad \text{for } x \in (0,\,1) \\ -u'(0) &= g_0 - \alpha_0\,u(0) \\ u(1) &= g_1 \end{cases}$$

In particular, specify the spaces V, V_0, and V_g, the bilinear form $a(\cdot,\,\cdot)$, and the linear form $\langle f,\,\cdot\rangle$. *Note:* The boundary condition in (b) at $x = 0$ is called *Robin boundary condition*, or boundary condition of 3^{rd} kind. *Hint:* Perform integration by parts as usual, substitute $u'(0)$ due to the Robin boundary condition, and collect the bilinear and linear terms accordingly.

03 Let the sequence $(u_k)_{k\in\mathbb{N}}$ of functions be defined by

$$u_k(x) = \begin{cases} 2x & \text{for } x \in \left[0,\,\frac{1}{2} - \frac{1}{2k}\right]\,, \\ 1 - \frac{1}{2k} - 2k\left(x - \frac{1}{2}\right)^2 & \text{for } x \in \left(\frac{1}{2} - \frac{1}{2k},\,\frac{1}{2} + \frac{1}{2k}\right)\,, \\ 2(1-x) & \text{for } x \in \left[\frac{1}{2} + \frac{1}{2k},\,1\right]\,. \end{cases}$$

Show that $u \in C^1[0,\,1]$. Let u be defined by

$$u(x) = \begin{cases} 2x & \text{for } x \in \left[0,\,\frac{1}{2}\right]\,, \\ 2(1-x) & \text{for } x \in \left(\frac{1}{2},\,1\right]\,. \end{cases}$$

Find out if $u \in H^1(0,\,1)$ or not and justify your answer. Calculate $\|u_k - u\|_{H^1(0,\,1)}$ (or find a suitable bound for it) and show that

$$\lim_{k\to\infty} \|u_k - u\|_{H^1(0,\,1)} = 0\,.$$

Use these results to show that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C^1[0,\,1]$ with respect to the H^1-norm, but that there exists no limit in $C^1[0,\,1]$.

04 Derive the variational formulation

$$\text{find } u \in V_g: \quad a(u,\,v) = \langle f,\,v\rangle \qquad \forall v \in V_0 \tag{1.3}$$

of the pure Neumann boundary value problem

$$\begin{aligned} -u''(x) &= f(x) \qquad \text{for } x \in (0,\,1)\,, \\ u'(0) &= g_0\,, \\ -u'(1) &= g_1\,, \end{aligned}$$

and show that the following statements are valid:

(a) If (1.3) has a solution, then

$$\langle f,\, c\rangle = 0, \qquad \forall c \in \mathbb{R}\,. \tag{1.4}$$

(b) If u is a solution of (1.3), then, for any constant $c \in \mathbb{R}$, $\widehat{u} := u + c$ is also a solution. *Hint:* Show and use $a(c,\, v) = 0$.

(c) If we choose $c = \overline{u} := \int_0^1 u(x)\, dx$, then

$$\widehat{u} \in \widehat{V} = \Big\{ v \in H^1(0,\, 1) : \int_0^1 v(x)\, dx = 0 \Big\}$$

(d) If $\widehat{u} \in \widehat{V}$ solves the variational problem

$$a(\widehat{u},\, \widehat{v}) = \langle f,\, \widehat{v}\rangle \quad \forall v \in \widehat{V}\,,$$

and if the condition (1.4) holds, then $\widehat{u}$ solves also (1.3). *Hint:* Each test function $v \in H^1(0,\, 1)$ can be written as $v(x) = \widehat{v}(x) + \overline{v}$ with $\overline{v} = \int_0^1 v(x)\, dx$ and $\widehat{v} \in \widehat{V}$.

05 Justify that

$$u(x) = \sqrt{2x - x^2}$$

is a *classical* solution from $C^2(0,\, 1) \cap C^1(0,\, 1] \cap C[0,\, 1)$ of the boundary value problem

$$-\big(a(x)\, u'(x)\big)' = 1 \qquad \text{for } x \in (0,\, 1)\,, \tag{1.5}$$

$$u(0) = 0\,, \tag{1.6}$$

$$a(1)\, u'(1) = 0\,, \tag{1.7}$$

where $a(x) = \sqrt{2x - x^2}$. Show that

$$\int_0^1 |u'(x)|^2\, dx = \infty\,.$$

Note: This example shows that $u \notin H^1(0,\, 1)$, i. e., u is no *weak* solution.

06 Let the coefficient $a \in L_\infty(0,\, 1)$ be defined by

$$a(x) = \begin{cases} a_1 & \text{for } x \in \big[0,\, \tfrac{1}{2}\big]\,, \\ a_2 & \text{for } x \in \big(\tfrac{1}{2},\, 1\big]\,, \end{cases}$$

with positive constants $a_1 \neq a_2$. Derive a variational formulation for the boundary value problem

$$\begin{aligned} -a(x)\, u''(x) &= f(x) \qquad \text{for } x \in (0,\, 1) \setminus \{\tfrac{1}{2}\}\,, \\ u(0) &= g_1\,, \\ u(1) &= g_2\,, \end{aligned}$$

with the *transmission conditions*

$$\begin{aligned} u(\tfrac{1}{2}-) &= u(\tfrac{1}{2}+)\,, \\ a_1\, u'(\tfrac{1}{2}-) &= -a_2\, u'(\tfrac{1}{2}+)\,, \end{aligned}$$

where, $w(\tfrac{1}{2}-)$ and $w(\tfrac{1}{2}+)$ denote the left sided and right sided limit of a function w, respectively. *Hint:* Integration by parts is only valid on the sub-intervals, separately.