Example 5.2: Double Layer potential operator V given on a circle Ω with the radius r, i.e.

$$\Gamma = \partial\Omega := \left\{ x = x(t) := r \begin{bmatrix} \cos 2\pi t \\ \sin 2\pi t \end{bmatrix} \in \mathbb{R}^2 : 0 \le t < 1 \right\}.$$

Then we have

$$E(x,y) = -\frac{1}{2\pi} \ln |x-y|$$

$$\nabla_x E(x,y) = -\frac{1}{2\pi} \frac{x-y}{|x-y|^2}$$

$$n_x = \frac{1}{|\dot{x}(t)|} \begin{pmatrix} \dot{x}_2(t) \\ -\dot{x}_1(t) \end{pmatrix} = \frac{1}{r} x(t)$$

$$Ku(y) = \int_\Gamma \frac{\partial E}{\partial n_x}(x,y)\, u(x)\, ds_x =$$

$$= -\frac{1}{2\pi r} \int_\Gamma \frac{(x, x-y)}{|x-y|^2} u(x)\, ds_x$$

$$|\dot{x}(t)| = 2\pi r$$

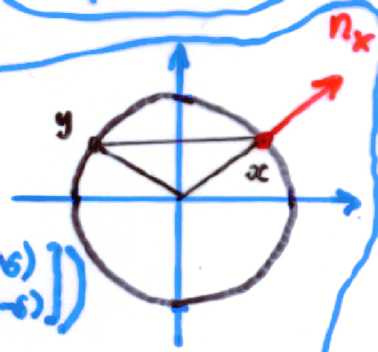
$$|x-y|^2 = |x(t) - y(s)|^2 =$$

$$= 4 r^2 \sin^2 \pi (t-s)$$

$$(x, x-y) = r^2 \left(\begin{bmatrix} \cos 2\pi t \\ \sin 2\pi t \end{bmatrix}, \begin{bmatrix} -2 \sin \pi (t+s) \sin \pi (t-s) \\ 2 \cos \pi (t+s) \sin \pi (t-s) \end{bmatrix} \right)$$

$$= \dots$$

$$\frac{(x, x-y)}{|x-y|^2} = \frac{1}{2} \quad \text{(mms)}$$



$$= -\frac{1}{2\pi r} \int_0^1 \frac{(x(t), x(t) - y(s))}{|x(t) - y(s)|^2} u(x(t))\, |\dot{x}(t)|\, dt$$

$$= -\frac{1}{2} \int_0^1 u(t)\, dt$$

Therefore, the EVP

$$K v_k = \lambda_k v_k, \quad k \in \mathbb{Z}$$

has the following solution (mms):

- eigenfunctions: $v_k = \exp(i 2\pi k t)$
- eigenvalues: $\lambda_k = \begin{cases} -1/2, & k = 0, \\ 0, & k \neq 0. \end{cases}$