Example 5.1: Single layer potential operator V given on a circle Ω with the radius r, i.e.

$$\Gamma = \partial\Omega := \left\{ x = x(t) := r \begin{bmatrix} \cos 2\pi t \\ \sin 2\pi t \end{bmatrix} \in \mathbb{R}^2 : 0 \le t < 1 \right\}.$$

Then we have:

$$E(x,y) = -\frac{1}{2\pi} \ln|x-y|$$

$$\underset{\overset{\shortparallel}{x(s)=y(s)}}{Vv(y)} = \int_\Gamma E(x,y) v(x)\, ds_x = -\frac{1}{2\pi} \int_\Gamma \ln|x-y|\, v(x)\, ds_x$$

$$= -\frac{1}{2\pi} \int_0^1 \ln|x(t) - \underset{\overset{\shortparallel}{y(s)}}{x(s)}|\, v(x(t))\, |\dot{x}(s)|\, ds$$

$$x(t) = r\begin{bmatrix} \cos 2\pi t \\ \sin 2\pi t \end{bmatrix}, \quad y(s) = r\begin{bmatrix} \cos 2\pi s \\ \sin 2\pi s \end{bmatrix}$$

$$|\dot{x}(t)| = \left| 2\pi r \begin{bmatrix} -\sin 2\pi t \\ \cos 2\pi t \end{bmatrix} \right| = 2\pi r \sqrt{\sin^2(\cdot) + \cos^2(\cdot)} = 2\pi r$$

$$|x(t) - y(s)| = r \left| \begin{bmatrix} \cos 2\pi t - \cos 2\pi s \\ \sin 2\pi t - \sin 2\pi s \end{bmatrix} \right| = r \left| \begin{matrix} -2\sin\pi(t+s)\sin\pi(t-s) \\ 2\cos\pi(t+s)\sin\pi(t-s) \end{matrix} \right|$$

$$= \sqrt{4r^2 \sin^2\pi(t-s)\,(\sin^2\pi(t+s) + \cos^2\pi(t+s))}$$

$$= \sqrt{4r^2 \sin^2\pi(t-s)} = |2r \sin\pi(t-s)|$$

$$= -\frac{1}{2\pi} \int_0^1 \ln|2r\sin\pi(t-s)|\, v(x(t))\, 2\pi r\, dt$$

$$= -r \ln 2r \int_0^1 v(x(t))\, dt - r \int_0^1 \ln|\sin\pi(t-s)|\, v(x(t))\, dt$$

Now we can conclude (mmb) that the functions

$$v_k(t) = v_k(x(t)) = \exp(i 2\pi k t), \quad k \in \mathbb{Z}$$

are obviously the eigenfunctions of the operator V:

$$V v_k = \lambda_k v_k \quad \text{with} \quad \lambda_k = \begin{cases} -r \ln r, & k = 0, \\ \dfrac{r}{2|k|}, & k \neq 0. \end{cases}$$

For $r > 1$ the eigenvalue $\lambda_0 < 0$ and V is <u>not</u> positive definite in this case!