T U T O R I A L

"Computational Mechanics"

to the lecture

"Numerical Methods in Continuum Mechanics 1"

Tutorial 09 Friday, May 30, 2008 (Time : $8^{\underline{30}}$ – $9^{\underline{15}}$ Room : SR T 1010)

4.3 Uzawa Algorithm and Schur Complement

32* Consider the mixed variational problem: Find $(u, \lambda) \in X \times \Lambda$ such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \langle F\,,\,v\rangle \quad \forall v \in X\,, \\ b(u,\mu) &= \langle G\,,\,\mu\rangle \quad \forall \mu \in \Lambda\,, \end{aligned}$$

where $F \in X^*$ and $G \in \Lambda^*$ are given. Let $A : X \to X^*$ and $B : X \to \Lambda^*$ be the related operators to $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, and let the assumptions of Theorem 2.4 (*Brezzi*) be satisfied. Show that the bilinearform

$$l(\xi,\eta) := \langle L\,\xi\,,\,\eta\rangle$$

with

$$L := \begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix}, \; \xi := \begin{pmatrix} u \\ \lambda \end{pmatrix}, \; \eta := \begin{pmatrix} v \\ \mu \end{pmatrix}, \text{ and } \|\xi\|_{X\times\Lambda} = \left(\|u\|_X^2 + \|\lambda\|_\Lambda^2\right)^{1/2}$$

satisfies the assumptions of Theorem 1.5 (*Babuska-Aziz*), if $a(\cdot,\cdot)$ is elliptic on the whole space X, i. e., if there exists $\alpha_1 > 0$ such that $a(v,v) \geq \alpha_1 \|v\|_X^2$ for all $v \in X$.

Hint: The LBB-condition

$$\exists \mu_1 > 0 \; \forall \xi = (u,\lambda) \in X \times \Lambda : \; \sup_\eta \frac{l(\xi,\eta)}{\|\eta\|} \geq \mu_1 \|\xi\|$$

can be shown by choosing $\eta = (v,\mu)$ such that $\mu = -2\lambda$, and $v = u + w$ where $w \in X$ is the solution of the adjoint problem $a(y,w) = b(y,\lambda)$ for all $y \in X$.

33* Consider the assumptions and definitions in Example 28 and replace $M = (-1,1) \times (-1,1)$ by $M_h = (-h,h) \times (-h,h)$, where $h \in (0,1]$. Show, that there exists a constant $c > 0$ independent of h ($c \neq c(h)$) such that

$$\|v_h\|_{H^1(M_h)} \leq c\,\|v\|_{H^1(M_h)} \quad \forall v \in C^1(\overline{M}_h) \; \forall h \in (0,1]\,.$$

Hint: Use $\|v_h\| \leq \|v_h - v\| + \|v\|$ and (after a proper transformation of variables) the estimate of Example 30.

$\boxed{34}$ Let $A \in R^{n\times n}$, $B \in R^{m\times n}$, and $C \in R^{m\times m}$ such, that $A = A^T$, $A > 0$, $C = C^T$, $C \geq 0$, and $\operatorname{Rank} B = \min\{m, n\}$. Show, that

1. the matrix $\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}$ is symmetric but indefinite,
2. the Schur complement matrix $S = BA^{-1}B^T + C$ is symmetric and positive definite ("SPD"),
3. if $C > 0$, then the matrix $\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix}$ is positive definite.

$\boxed{35}$ Write down the Uzawa–CG Algorithm, i.e., the CG Algorithm for the system

$$\text{Given } \underline{f} \in \mathbb{R}^n \text{ and } \underline{g} \in \mathbb{R}^m. \text{ Find } \underline{\lambda} \in \mathbb{R}^m : \quad \left(BA^{-1}B^T + C\right)\underline{\lambda} = BA^{-1}\underline{f} - \underline{g}\,.$$