T U T O R I A L

“Computational Mechanics”

to the lecture

“Numerical Methods in Continuum Mechanics 1”

Tutorial 03 Friday, April 11, 2008 (Time : $8^{\underline{30}} - 9^{\underline{15}}$ Room : SR T 1010)

2 Theorem of Babuska and Aziz

Theorem 1.5: (Babuška und Aziz, 1972)
Let U and V be Hilbert spaces. Then the linear map (operator) $A : U \longmapsto V^*$ is an isomorphism (bijective, A and A^{-1} continuous) if and only if (= iff) the corresponding bilinear form $a(.,.) : U \times V \to R^1$ fulfills the following conditions:

1. continuity, i.e. $\exists\, \mu_2 = \text{const.} > 0$:

$$(8) \quad |a(u,v)| \leq \mu_2 \|u\|_U \|v\|_V \ \forall u \in U, \forall v \in V,$$

2. inf-sup-condition, i.e. $\exists\, \mu_1 = \text{const.} > 0$:

$$(9) \quad \inf_{u \in U} \sup_{v \in V} \frac{a(u,v)}{\|u\|_U \|v\|_V} \geq \mu_1 = \text{const.} > 0,$$

3. $\forall v \in V \setminus \{0\}\ \exists u \in U$:

$$(10) \quad \text{a(u,v)} \neq 0.$$

Proof:

1. $\Longleftarrow$ **sufficiency**, i.e (8) - (9) $\Longrightarrow A$ is an isomorphism !

   - (8) $\Longleftrightarrow A : U \mapsto V^*$ - continuous (= bounded), i.e. $A \in L(U, V^*)$ and $\|A\|_{L(U,V^*)} \leq \mu_2$.
   - A is injective (= one-to-one mapping). Indeed, assuming $Au_1 = Au_2 \Longleftrightarrow a(u_1, v) = a(u_1, v)\ \forall v \in V$ and using (9), we obtain

   $$0 = \sup_{v \in V} \frac{a(u_1 - u_2, v)}{\|v\|_V} \geq \mu_1 \|u_1 - u_2\|_U,$$

   i.e. $u_1 = u_2$.
   - $A : U \leftrightarrow A(U) \subset V^*$ yields $\exists A^{-1} : A(U) \mapsto U$.
   - $A^{-1} : A(U) \mapsto U$ is continuous. Indeed,

   $$\mu_1 \|u\|_U \leq \sup_{v \in V} \frac{< Au, v >}{\|v\|_V} = \sup_{v \in V} \frac{< f, v >}{\|v\|_V} = \|f\|_{V^*},$$

   i.e. $\mu_1 \|A^{-1} f\|_U \leq \|f\|_{V^*}\ \forall f \in A(U)$,
   i.e. A^{-1} is bounded, i.e. A^{-1} is continuous.

- $A : U \leftrightarrow A(U),\ A^{-1} : A(U) \leftrightarrow U$ - continuous $\Longrightarrow A(U) = \overline{A(U)}$.
- Closed Range Theorem (Theorem 1.4) yields: $A(U) = \overline{A(U)} = (\ker A^*)^0 = \{0\}^0 = V^*$ due to (10). Indeed,
  $\ker A^* = \{v \in V : < A^*v, u >_{U^* \times U} = < Au, v >_{V^* \times V} = a(u, v) = 0\, \forall u \in U\} = \{0\}$.

An alternative proof of the sufficiency follows from Exercises 13*. This results in a constructive proof !

2. $\Longrightarrow$ **necessity**: see Exercises 10 - 12 !

10 Show that the fact that $A \in L(U, V^*)$ is an isomorphism yields the continuity (= boundedness) of the bilinear form $a(u, v) =< Au, v >$! Define the boundedness constant μ_2 of the bilinear form $a(.,.)$!

11 Show that the fact that $A \in L(U, V^*)$ is an isomorphism yields the inf-sup-condition of the bilinear form $a(u, v) =< Au, v >$! Define the inf-sup-constant μ_1 of the bilinear form $a(.,.)$!

12 Show that the fact that $A \in L(U, V^*)$ is an isomorphism also yields the third condition of the Babuška-Aziz-Theorem 1.5 !

13* Let us assume that the sufficient conditions (8) - (9) of the Babuška-Aziz-Theorem 1.5 are fulfilled. Let us consider the variational problem: find $u \in U$ such that

$$A(u, v) =< F, v > \quad \forall v \in V, \tag{2.26}$$

where the bilinear form $A(u, v)$ and the linear form $< F, v >$ are defined by the identities

$$A(u, v) =< A^* J Au, v > \quad \forall u, v \in U \tag{2.27}$$

and

$$< F, v > =< A^* J f, v > \quad \forall v \in U, \tag{2.28}$$

respectively. Here $A^* : V \longrightarrow U^*$ denotes the adjoint to $A : U \longrightarrow V^*$ operator and $J : V^* \longrightarrow V$ is the Riesz isomorphism between the Hilbertspaces V^* and V. Show that the linear form $< F, . >$ and bilinear form $A(.,.)$ fulfil the assumption of the Lax-Milgram-Theorem and provide the ellipticity and the boundedness constants of the bilinear form $A(.,.)$!