Rigid Body Motions ($= \operatorname{Ker} \varepsilon(v)$):

- **Lemma 3.1:**

> Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\Gamma = \partial\Omega \in C^{0,1}$ and $v \in [H^1(\Omega)]^3$. Then
>
> $$\varepsilon(v) = 0 \Longleftrightarrow v \in \mathcal{R} := \{v(x) = a \times x + b : a, b \in \mathbb{R}^3\},$$
>
> where
>
> $$\mathcal{R} := \operatorname{span}\Big\{\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{translations}}, \underbrace{\begin{pmatrix}-x_2\\x_1\\0\end{pmatrix}, \begin{pmatrix}0\\-x_3\\x_2\end{pmatrix}, \begin{pmatrix}-x_3\\0\\x_1\end{pmatrix}}_{\text{rotations}}\Big\} \subset [H^1(\Omega)]^3$$
>
> denotes the subspace of rigid body motions

- **Proof:**

a) $v \in \mathcal{R} \Longrightarrow \varepsilon_{ij}(v) = 0 \;\forall i,j = \overline{1,3}$ (trivial)

b) $\varepsilon_{ij}(v) = 0 \;\forall i,j = \overline{1,3} \Longrightarrow v \in \mathcal{R}$

$$v_{k,ij} = \underbrace{\varepsilon_{jk,i} + \varepsilon_{ik,j} - \varepsilon_{ij,k}}_{\frac{1}{2}(\cancel{v_{j,ki}} + v_{k,ji} + \cancel{v_{i,kj}} + v_{k,ij} - \cancel{v_{i,jk}} - \cancel{v_{j,ik}})} \overset{\varepsilon_{ij}(v)=0}{=} 0 \quad \text{in } H^{-1}(\Omega)$$

$\forall i,j = \overline{1,3} \quad \forall k = \overline{1,3}$

$\Rightarrow \; v(x) = Ax + b = [a_{ij}x_j + b_i]$ – affine linear,

with A – (3×3)-matrix, $b \in \mathbb{R}^3$.

$\Rightarrow \; \varepsilon(v) = 0 \Rightarrow a_{kk} = 0$ ($\cancel{\Sigma}$) and $a_{kl} = -a_{lk}$,

i.e. $A = -A^T = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$

$\Leftrightarrow \; v(x) = a \times x + b \doteq \mathcal{R}$

- **Exercise 3.2:** 2D, i.e. $\Omega \subset \mathbb{R}^2$ – Υ, Lip

$\varepsilon_{ij}(v) = 0$, $i,j = \overline{1,2}$, $v \in [H^1(\Omega)]^2 \Leftrightarrow v \in \mathcal{R} = \,?$