- Finally, we prove the estimate from above in (63):

$$[T(\underline{u},\underline{\Delta}),(\underline{u},\underline{\Delta})] \overset{(iii)}{=} \underbrace{[T(\underline{u}_0,\underline{0}),(\underline{u}_0,\underline{0})]} + \underbrace{[T(\underline{u}_H,\underline{\Delta}),(\underline{u}_H,\underline{\Delta})]}$$

$$[T(\underline{u}_H,\underline{\Delta}),(\underline{u}_H,\underline{\Delta})] \overset{(ii)}{=} ((BA^{-1}B^T+C)\underline{\Delta},\underline{\Delta})$$

$$\begin{aligned}
[T(\underline{u}_0,\underline{0}),(\underline{u}_0,\underline{0})] &= ((A-A_0)A_0^{-1}A\,\underline{u}_0,\underline{u}_0) \\
&= \underbrace{((A-A_0)A_0^{-1}(A-A_0)\underline{u}_0,\underline{u}_0)} + ((A-A_0)\underline{u}_0,\underline{u}_0)
\end{aligned}$$

$$((A-A_0)A_0^{-1}(A-A_0)\underline{u}_0,\underline{u}_0) = (A_0^{-1}(A-A_0)\underline{u}_0,(A-A_0)\underline{u}_0) \le \frac{\alpha}{1-\alpha}((A-A_0)\underline{u}_0,\underline{u}_0)$$

$$\boxed{\begin{aligned}
A_0^{-1} &\le \frac{\alpha}{1-\alpha}(A-A_0)^{-1} \\
(1-\alpha)A_0^{-1} &\le \alpha(A-A_0)^{-1} \\
(1-\alpha)(A-A_0) &\le \alpha A_0 \\
A-A_0 &\le \alpha A
\end{aligned}}$$

$$\begin{aligned}
&\le \frac{1}{1-\alpha}\underbrace{((A-A_0)\underline{u}_0,\underline{u}_0)} \\
&((A-A_0)\underline{u}_0,\underline{u}_0) = ((A-A_0)(\underline{u}-\underline{u}_H),\underline{u}-\underline{u}_H) \\
&\le (1+\varepsilon)((A-A_0)\underline{u},\underline{u}) + \left(1+\frac{1}{\varepsilon}\right)\underbrace{((A-A_0)\underline{u}_H,\underline{u}_H)} \\
&((A-A_0)\underline{u}_H,\underline{u}_H) \overset{(58)}{\le} \alpha(A\underline{u}_H,\underline{u}_H) \overset{(i)}{=} \alpha(BA^{-1}B^T\underline{\Delta},\underline{\Delta})
\end{aligned}$$

$$\le \underbrace{\frac{1+\varepsilon}{1-\alpha}}((A-A_0)\underline{u},\underline{u}) + \Big(\underbrace{1+\frac{\alpha}{1-\alpha}\left(1+\frac{1}{\varepsilon}\right)}\Big)((BA^{-1}B^T+C)\underline{\Delta},\underline{\Delta})$$

$$\frac{1+\varepsilon}{1-\alpha} = 1+\frac{\alpha}{1-\alpha}\left(1+\frac{1}{\varepsilon}\right) \Leftarrow \begin{array}{l} 1+\varepsilon = 1-\alpha+\alpha\left(1+\frac{1}{\varepsilon}\right) \\ \varepsilon^2=\alpha,\ \text{i.e. } \varepsilon=\sqrt{\alpha} \end{array}$$

$\varepsilon = \sqrt{\alpha}$ $\downarrow$

$$= \frac{1+\sqrt{\alpha}}{1-\alpha}\Big[((A-A_0)\underline{u},\underline{u}) + ((BA^{-1}B^T+C)\underline{\Delta},\underline{\Delta})\Big] = \gamma\,[\tilde{T}\underline{X},\underline{X}]$$

q.e.d.