1.1.1. Primal Variational Formulation and Finite Element Approximation

- **Primal Variational Formulation:**

(1) Find $u \in V_g \underset{\uparrow}{=} V_0 : a(u,v) = \langle f, v\rangle \; \forall v \in V_0$

homogenization $\quad Au = f$ in V_0^*

- **Main Result:** = the Lax-Milgram theorem

= Banach's fixed point theorem

= Theorem 1.7 + Corollary 1.8 (NuPDE)

> **Ass.:** 0. $V_0 \subset V$ - subspace of the H-space $V, \|\cdot\|_V, (\cdot,\cdot)_V$
> 1. $f \in V_0^*$
> 2. $a(\cdot,\cdot) : V_0 \times V_0 \mapsto \mathbb{R}^1$ - bilinear form:
> 2a) V_0-elliptic: $\mu_1 \|v\|_V^2 \le a(v,v) \; \forall v \in V_0$
> 2b) V_0-bounded: $|a(u,v)| \le \mu_2 \|u\|_V \|v\|_V \; \forall v \in V_0$
>
> **St.:** $\exists!\, u \in V_0$: (1) and fixed point iteration.

Proof: $Au = f \iff u = u - \tau(JAu - Jf) =: \Phi_\tau(u)$

Fixed point iteration: $u^{n+1} = \Phi(u^n) \xrightarrow{n\to\infty} u$ in V_0

$\tau \in (0, 2\mu_1/\mu_2^2)$

A priori estimates:

$\|u - u^{n+1}\| \le q \|u - u^n\| \le q^{n+1} \|u - u^0\|$

$q = q(\tau) = \sqrt{1 - 2\mu_1\tau + \mu_2^2\tau^2} < 1$

$\vee\!|$

A posteriori estimates:

$q(\tau_{opt}) = q_{opt} = \sqrt{1 - \left(\frac{\mu_1}{\mu_2}\right)^2}$

$\|u - u^{n+1}\| \le \frac{q^{n+1}}{1-q} \|u_1 - u_0\|$

$\|u - u^{n+1}\| \le \frac{q}{1-q} \|u_{n+1} - u_n\|$

$\tau_{opt} = \mu_1 / \mu_2^2$