ÜBUNGEN ZU

NUMERIK ELLIPTISCHER PROBLEME

für den 24. 6. 2008

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55. Let $N \in \mathbb{N}$ and set $H = 1/N$, $n = 2N$, $h = 1/n$. Let

$$\overline{\omega}_h = \{x_0, x_1, \dots, x_n\} \quad \text{and} \quad \overline{\omega}_H = \{X_0, X_1, \dots, X_N\}$$

with $x_i = i\,h$ and $X_i = i\,H = 2i\,h = x_{2i}$. Let V_{0h} denote the set of grid functions $v_h : \overline{\omega}_h \longrightarrow \mathbb{R}$ which vanish at $x_0 = 0$ and $x_n = 1$ and let V_{0H} denote the set of grid functions $v_H : \overline{\omega}_H \longrightarrow \mathbb{R}$ which vanish at $X_0 = 0$ and $X_N = 1$.

The grid functions $\varphi_{h,k} \in V_{0h}$, $k = 1, \dots, n-1$, are given by

$$\varphi_{h,k}(x_i) = \sqrt{2h}\, \sin(k\pi x_i) \quad i = 1, \dots, n-1.$$

The basis functions $\varphi_{H,k} \in V_{0H}$, $k = 1, \dots, N-1$, are given by

$$\varphi_{H,k}(X_i) = \sqrt{2H}\, \sin(k\pi X_i) \quad i = 1, \dots, N-1.$$

The difference operator $L_h : V_{0h} \longrightarrow V_{0h}$ be given by

$$(L_h v_h)(x_i) = \frac{1}{h}\left[-v_h(x_{i-1}) + 2\,v_h(x_i) - v_h(x_{i+1})\right] \quad \text{for all } i = 1, \dots, n-1.$$

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Show: If

$$z_h = \sum_{k=1}^{n-1} \alpha_k\, \varphi_{h,k},$$

then

$$L_h z_h = \sum_{k=1}^{n-1} \alpha'_k\, \varphi_{h,k}$$

with

$$\alpha'_k = \lambda_{h,k}\, \alpha_k \quad \text{with } \lambda_{h,k} = \frac{4}{h}\, \sin^2(k\pi h/2) \quad \text{for all } k = 1, \dots, n-1.$$

Hint: Show first $L_h \varphi_{h,k} = \lambda_{h,k}\, \varphi_{h,k}$ by using the identity $\sin(x-y) + \sin(x+y) = 2 \sin x \cos y$.

56. The restriction $I_h^H : V_{0h} \longrightarrow V_{0H}$ is given by

$$(I_h^H r_h)(X_i) = \frac{1}{2} r_h(x_{2i-1}) + r_h(x_{2i}) + \frac{1}{2} r_h(x_{2i+1}).$$

Show:

$$\begin{aligned} I_h^H \varphi_{h,k} &= \sqrt{2}\, \cos^2(k\pi h/2)\, \varphi_{H,k} \quad \text{for all } k = 1, \ldots, N-1, \\ I_h^H \varphi_{h,N} &= 0, \\ I_h^H \varphi_{h,n-k} &= -\sqrt{2}\, \sin^2(k\pi h/2)\, \varphi_{H,k} \quad \text{for all } k = 1, \ldots, N-1 \end{aligned}$$

Hint for the last identity: Show first: $\varphi_{h,n-k}(x_i) = -(-1)^i \, \varphi_{h,k}(x_i)$.

57. Show: If

$$r_h = \sum_{k=1}^{n-1} \beta_k \, \varphi_{h,k},$$

then

$$I_h^H r_h = \sum_{k=1}^{N-1} \beta_k' \, \varphi_{H,k}$$

with

$$\beta_k' = \sqrt{2}\, \left[\cos^2(k\pi h/2)\beta_k - \sin^2(k\pi h/2)\beta_{n-k}\right].$$

58. Show: If

$$r_H = \sum_{k=1}^{N-1} \gamma_k \, \varphi_{H,k},$$

then

$$w_H = \sum_{k=1}^{N-1} \gamma_k' \, \varphi_{H,k}$$

with

$$\gamma_k' = \frac{1}{\lambda_{H,k}} \gamma_k, \quad \text{where } \lambda_{H,k} = \frac{4}{H} \sin^2(k\pi H/2), \quad \text{for all } k = 1, \ldots, N-1$$

satisfies the equation

$$L_H w_H = r_H.$$

59. The prolongation operator $I_H^h : V_{0H} \longrightarrow V_{0h}$ is given by

$$\begin{aligned} (I_H^h w_H)(x_{2i}) &= w_H(X_i) \quad \text{for all } i = 1, \ldots, N-1 \\ (I_H^h w_H)(x_{2i-1}) &= \frac{1}{2} \left[w_H(X_{i-1}) + w_H(X_i)\right] \quad \text{for all } i = 1, \ldots, N \end{aligned}$$

Show:

$$I_H^h \varphi_{H,k} = \sqrt{2}\, \left[\cos^2(k\pi h/2)\, \varphi_{h,k} - \sin^2(k\pi h/2)\, \varphi_{h,n-k}\right] \quad \text{for all } k = 1, \ldots, N-1.$$

60. Show: If

$$w_H = \sum_{k=1}^{N-1} \delta_k \, \varphi_{H,k},$$

then

$$I_H^h w_H = \sum_{k=1}^{n-1} \delta'_k \, \varphi_{h,k}$$

with

$$\begin{aligned} \delta'_k &= \sqrt{2}\, \cos^2(k\pi h/2)\, \delta_k \quad \text{for all } k = 1, \ldots, N-1, \\ \delta'_N &= 0, \\ \delta'_{n-k} &= -\sqrt{2}\, \sin^2(k\pi h/2)\, \delta_k \quad \text{for all } k = 1, \ldots, N-1. \end{aligned}$$