ÜBUNGEN ZU

NUMERIK ELLIPTISCHER PROBLEME

für den 17. 6. 2008

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49. Let $\mathcal{T}_h = \{\delta_r \colon r \in \mathbb{R}_h\}$ be an admissible subdivision of a polygonal domain $\Omega \subset \mathbb{R}^2$ into acute triangles and let $\mathcal{T}_{\mathcal{H}} = \{\mathcal{H}(x) \colon x \in \overline{\omega}_h\}$ be the secondary mesh obtained by the PB method.

    The Petrov-Galerkin method discussed in class for the boundary value problem

$$\begin{aligned} -\operatorname{div}(a(x)\operatorname{grad} u(x)) &= f(x) \qquad x \in \Omega, \\ u(x) &= 0 \qquad x \in \partial\Omega \end{aligned}$$

    leads to the discrete variational problem:

    Find $u_h \in V_{0h}$ such that

$$\bar{a}(u_h, \bar{v}) = (f, \bar{v})_{L^2(\Omega)} \quad \forall \bar{v} \in T_{0h}$$

    (with the notations introduced in class).

    Show: This discrete variational problem can be written as a finite difference method

$$\begin{aligned} (L_h \underline{u}_h)(x) &= f_h(x) \qquad x \in \omega_h, \\ \underline{u}_h(x) &= 0 \qquad x \in \gamma_h \end{aligned}$$

    for the grid function $\underline{u}_h \colon \overline{\omega}_h \to \mathbb{R}$, given by $\underline{u}_h(x) = u_h(x)$ for all $x \in \overline{\omega}_h$, with

$$(L_h v)(x) = -\frac{1}{H(x)} \sum_{\xi \in S_h'(x)} \bar{a}(x_\xi) \frac{v(\xi) - v(x)}{h(x,\xi)} s(x_\xi) \quad \text{for all } x \in \omega_h$$

    and

$$\bar{a}(x_\xi) = \frac{1}{s(x_\xi)} \int_{\zeta(x,\xi)} a(y)\, ds, \quad f_h(x) = \frac{1}{H(x)} \int_{\mathcal{H}(x)} f(y)\, dy.$$

50. Assume the notations introduced in class and the previous example.

    Let the discrete Laplace operator Δ_h be given by

$$(\Delta_h v)(x) = \frac{1}{H(x)} \sum_{\xi \in S_h'(x)} \frac{v(\xi) - v(x)}{h(x,\xi)} s(x_\xi), \quad \text{for all } x \in \omega_h.$$

    The following discrete scalar products are introduced for grid functions: $v \colon \overline{\omega}_h \to \mathbb{R}$, $w \colon \overline{\omega}_h \to \mathbb{R}$ with $v(x) = w(x) = 0$ for all $x \in \gamma_h$:

$$\begin{aligned} (v,w)_{L^2(\omega_h)} &= \sum_{x \in \omega_h} v(x) w(x) H(x), \\ (v,w)_{H_0^1(\omega_h)} &= \sum_{x_\xi} \frac{[v(\xi) - v(x)][w(\xi) - w(x)]}{h(x,\xi)^2} H'(x_\xi). \end{aligned}$$

Show the identity

$$(-\Delta_h v, w)_{L^2(\omega_h)} = (v, w)_{H_0^1(\omega_h)}.$$

51. Assume the notations introduced in class and the previous examples.

    Let $w_h \in V_{0h}$, then $\underline{w}_h$ denotes the corresponding grid function $\underline{w}_h \colon \overline{\omega}_h \to \mathbb{R}$, given by $\underline{w}_h(x) = w_h(x)$ for all $x \in \overline{\omega}_h$.

    Show the identity:

    $$(\underline{u}_h, \underline{v}_h)_{H_0^1(\omega_h)} = (\operatorname{grad} u_h, \operatorname{grad} v_h)_{L^2(\Omega)} \quad \forall u_h, v_h \in V_{0h}.$$

    Hint:

    (a) Express $(\underline{u}_h, \underline{v}_h)_{H_0^1(\omega_h)}$ with the help of $\Delta_h \underline{u}_h$, see example 50.

    (b) Express $\Delta_h \underline{u}_h$ with the help of the bilinear form $\bar{a}$, see example 49.

    (c) Express $\bar{a}$ with the help of the bilinear form a, given by

    $$a(u, v) = (\operatorname{grad} u, \operatorname{grad} v)_{L^2(\Omega)},$$

    see class.

52. Assume the notations introduced in class and the previous examples.

    Let $w_h \in V_{0h}$, then $\underline{w}_h$ denotes the corresponding grid function $\underline{w}_h \colon \overline{\omega}_h \to \mathbb{R}$, given by $\underline{w}_h(x) = w_h(x)$ for all $x \in \overline{\omega}_h$, and $\overline{w}_h \in T_{0h}$ denotes the corresponding piecewise constant function (with respect to the secondary grid) with $\overline{w}_h(x) = w_h(x)$ for all $x \in \overline{\omega}_h$.

    Show the identity

    $$(\underline{u}_h, \underline{v}_h)_{L^2(\omega_h)} = (\overline{u}_h, \overline{v}_h)_{L^2(\Omega)} \quad \forall u_h, v_h \in V_{0h}.$$

53. Assume the notations introduced in class and the previous examples.

    Show that there exist constants $\underline{c}, \overline{c} > 0$, independent of h, such that

    $$\underline{c}\,(\overline{v}_h, \overline{v}_h)_{L^2(\Omega)} \leq (v_h, v_h)_{L^2(\Omega)} \leq \overline{c}\,(\overline{v}_h, \overline{v}_h)_{L^2(\Omega)} \quad \forall u_h, v_h \in V_{0h}.$$

    Hint: Transform to the reference triangle and observe that

    $$\lambda_{\min}(\hat{M})(\hat{v}, \hat{v})_{\ell^2} \leq (\hat{M}\hat{v}, \hat{v})_{\ell^2} \leq \lambda_{\max}(\hat{M})(\hat{v}, \hat{v})_{\ell^2},$$

    where $\hat{M}$ denotes the mass matrix on the reference triangle.

54. Assume the notations introduced in class and the previous examples.

Show the discrete Friedrichs inequality: There exists a constant $\widetilde{c}_F > 0$, independent of h, such that

$$\|\underline{v}\|_{L^2(\omega_h)} \leq \widetilde{c}_F \, |\underline{v}|_{H^1(\omega_h)}$$

for all grid functions $\underline{v} \colon \overline{\omega}_h \to \mathbb{R}$ with $\underline{v}(x) = 0$ for all $x \in \gamma_h$. You can use the (continuous) Friedrichs inequality

$$\|v\|_{L^2(\Omega)} \leq c_F \, |v|_{H^1(\Omega)} \quad \text{for all } v \in H_0^1(\Omega)$$

without proof.