ÜBUNGEN ZU

NUMERIK ELLIPTISCHER PROBLEME

für den 10. 6. 2008

43. Consider the boundary value problem:

$$Lu(x) = f(x) \quad \forall x \in (0, 1),$$
$$u(0) = u(1) = 0$$

with

$$Lu(x) = -u''(x) + bu'(x)$$

and $b \in \mathbb{R}$. Let \mathcal{T}_h denote the equidistant subdivision of the interval [0, 1] with mesh size $h = 1/n, n \in \mathbb{N}$, the corresponding set of nodes is given by $\overline{\omega}_h = \{x_0, x_1, \ldots, x_n\}$. Consider the following variational formulation of the boundary value problem: Find $u \in H_0^1(0, 1)$ such that

$$a(u,v) = (f,v)_{L^2(0,1)} \quad \forall v \in H^1_0(0,1)$$
(1)

with

$$a(u, v) = (u', v')_{L^2(0,1)} + (bu', v)_{L^2(0,1)}$$

For b = 0, the finite element discretization by the Courant element is equivalent to the finite difference method

$$(L_h u_h)(x) = f_h(x) \quad \forall x \in \omega_h,$$
$$u_h(0) = u_h(1) = 0$$

with

$$(L_h u_h)(x) = -\frac{1}{h^2} \left[u_h(x-h) - 2u_h(x) + u_h(x+h) \right]$$

and

$$f_h(x) = \frac{1}{h} \int_{x-h}^{x+h} f(y) \, p^{(x)}(y) \, dy,$$

where $p^{(x)}(y)$ denotes the nodal basis function associated to the node $x \in \overline{\omega}_h$. This follows from the identity

$$\frac{1}{h}a(u_h, p^{(x)}) = (L_h u_h)(x),$$

which is easy to show.

Which finite difference method is equivalent to the finite element discretization by the Courant element for the case $b \neq 0$?

44. Assume the notations of the previous example.

Let δ_h be a given positive real number and consider the following discrete variational problem: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h)_{L^2(0,1)} \quad \forall v_h \in V_h$$
(2)

with

$$a_h(u_h, v_h) = (u'_h, v'_h)_{L^2(0,1)} + (bu'_h, v_h + \delta_h \, bv'_h)_{L^2(0,1)}$$

where $V_h \subset H_0^1(0, 1)$ denotes the finite element space given by the Courant element. Determine the parameter δ_h such that the finite element discretization by the Courant element applied to (2) is equivalent to the following finite difference method:

$$(L_h u_h)(x) = f_h(x) \quad \forall x \in \omega_h;$$
$$u_h(0) = u_h(1) = 0$$

with

$$(L_h u_h)(x) = -\frac{1}{h^2} [u_h(x-h) - 2u_h(x) + u_h(x+h)] + b \left\{ \begin{array}{l} \frac{1}{h} [u_h(x) - u_h(x-h)] & \text{if } b \ge 0 \\ \frac{1}{h} [u_h(x+h) - u_h(x)] & \text{if } b < 0 \end{array} \right\}.$$

45. A finite difference method of the form

$$(L_h u_h)(x) = f_h(x) \quad x \in \omega_h$$
$$u_h(0) = u_h(1) = 0$$

with

$$(L_h u_h)(x) = A_h(x)u_h(x) - \sum_{\xi \in S'_h(x)} B_h(x,\xi)u_h(\xi)$$

is called monotone if and only if

- (a) $A_h(x) > 0$ for all $x \in \omega_h$
- (b) $B_h(x,\xi) > 0$ for all $\xi \in S'_h(x), x \in \omega_h$.
- (c) $D_h(x) \equiv A(x) \sum_{\xi \in S'(x)} B(x,\xi) \ge 0$ for all $x \in \omega_h$.

Under what conditions on b and h are the finite difference methods of the two previous examples monotone?

46. Assume the notations of the previous examples.

Show that

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} a_T(u_h, v_h + \delta_h \, bv'_h).$$

with

$$a_T(w,v) = (u',v')_{L^2(T)} + (bu',v)_{L^2(T)}.$$

47. Assume the notations of the previous examples. Show that

$$a_h(u_h, v_h) = a(u_h, v_h) + \sum_{T \in \mathcal{T}_h} \delta_h(Lu_h, Lv_h)_{L^2(T)}.$$

48. Assume the notations of the previous examples.

Consider the following variational problem: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \langle F_h, v_h \rangle \quad \forall v_h \in V_h$$

with

$$a_h(u_h, v_h) = a(u_h, v_h) + \sum_{T \in \mathcal{T}_h} \delta_h(Lu_h, Lv_h)_{L^2(T)}$$

and

$$\langle F_h, v_h \rangle = (f, v_h)_{L^2(0,1)} + \sum_{T \in \mathcal{T}_h} \delta_h (f, Lv_h)_{L^2(T)}.$$

Let $u \in H_0^1(0,1)$ be the exact solution of (1). Show: If $u \in H_0^1(0,1) \cap H^2(0,1)$, then

$$\langle F_h, v_h \rangle - a_h(u, v_h) = 0 \quad \forall v_h \in V_h.$$