# ÜBUNGEN ZU

# NUMERIK ELLIPTISCHER PROBLEME

für den 3. 6. 2008

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37. Let $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ be an admissible triangulation (subdivision into triangles) of a domain $\Omega \subset \mathbb{R}^2$ (the primary grid). Let $\mathcal{E}_h$ be the set of edges of the triangulation. For each $e \in \mathcal{E}_h$, let $x_e$ denote the midpoint of the edge $e$. Consider the following secondary grid: For each interior edge $e$, the box $\mathcal{H}(x_e)$ is the quadrilateral whose vertices are the two endpoints of the edge $e$ and the centroids of the two triangles which share the common edge $e$. Complete the definition of the secondary grid by defining appropriate boxes associated to edges on the boundary of the domain $\Omega$ such that the following conditions are satisfied:

    (a) $\overline{\Omega} = \bigcup_{e \in \mathcal{E}_h} \overline{\mathcal{H}(x_e)}$.

    (b) $\mathcal{H}(x_e) \cap \mathcal{H}(x_f) = \emptyset$ for all $e, f \in \mathcal{E}_h$ with $e \neq f$.

    For each interior edge $e$, express the area $|\mathcal{H}(x_e)|$ of the box $\mathcal{H}(x_e)$ in terms of the area of the two triangles, say $\delta_r$ and $\delta_s$, which share the common edge $e$ and conclude that

    (c) There is a constant $c$ such that $|\mathcal{H}(x_e)| \leq c\, h^2$ for all interior edges $e$.

38. Assume the notations and assumptions from exercise 37.

    Consider the boundary value problem:

$$\begin{aligned} -\Delta u(x) &= f(x) \quad \text{in } \Omega, \\ u(x) &= 0 \qquad\; \text{on } \Gamma. \end{aligned}$$

    Show for $u \in C^2(\overline{\Omega})$ that

$$-\int_{\partial\mathcal{H}(x_e)} \frac{\partial u}{\partial n}(y)\; ds = \int_{\mathcal{H}(x_e)} f(y)\; dy,$$

    and

$$-\int_{\partial\mathcal{H}(x_e)} \frac{\partial u}{\partial n}(y)\; ds = -\int_{(\partial\delta_{e,r})\setminus e} \frac{\partial u}{\partial n}(y)\; ds - \int_{(\partial\delta_{e,s})\setminus e} \frac{\partial u}{\partial n}(y)\; ds,$$

    where $\delta_r$ and $\delta_s$ denote the triangles which share the common edge $e$, and $\delta_{e,r} = \mathcal{H}(x_e) \cap \delta_r$, $\delta_{e,s} = \mathcal{H}(x_e) \cap \delta_s$, for each interior edge $e$.

39. Assume the notations and assumptions from exercise 38.

    Let $u_h$ be a piece-wise linear function on $\overline{\Omega}$, i.e.: $u_h|_{\delta_r} \in P_1$ for all $r \in \mathbb{R}_h$.

Show that

$$-\int_{(\partial\delta_{e,r})\setminus e} \frac{\partial u_h}{\partial n}(y)\ ds - \int_{(\partial\delta_{e,s})\setminus e} \frac{\partial u_h}{\partial n}(y)\ ds = \left[\frac{\partial u_h}{\partial n}\right]_e (x_e)\,|e|,$$

where $|e|$ denotes the length of the edge $e$ and jump term on $e$ for $x \in e$ is defined by

$$\left[\frac{\partial u_h}{\partial n}\right]_e (x) = \lim_{t\downarrow 0} n_{e,r} \cdot \operatorname{grad} u_h(x - t\, n_{e,r}) + \lim_{t\downarrow 0} n_{e,s} \cdot \operatorname{grad} u_h(x - t\, n_{e,s}),$$

where $n_{e,r}$ ($n_{e,s}$) denotes the outward normal unit vector of the triangle $\delta_e$ ($\delta_s$) on the edge $e$.

Hint: Show and use

$$\int_{(\partial\delta_{e,r})\setminus e} \frac{\partial u_h}{\partial n}(y)\ ds = \underbrace{\int_{\partial\delta_{e,s}} \frac{\partial u_h}{\partial n}(y)\ ds}_{\int_{\delta_{e,s}} \Delta u_h(y)\ dy} - \int_e \frac{\partial u_h}{\partial n}(y)\ ds.$$

40. Assume the notations and assumptions from exercise 39. Additionally, assume that $u_h$ is continuous at $x_e$ for all $e \in \mathcal{E}_h$, i.e.: the value of $u_h$ at $x_e$ is well-defined for all $e \in \mathcal{E}_h$.

    Express $\lim_{t\downarrow 0} n_{e,r} \cdot \operatorname{grad} u_h(x - t\, n_{e,r})$ as a linear combination of the values of $u_h$ at $x_e$, $x_f$ and $x_g$, where $x_f$ and $x_g$ denote the other two midpoints of edges for the triangle $\delta_r$.

    Hint: Show and use on the triangle $\delta_r$:

$$\frac{\partial u_h}{\partial n}(x_e) = \frac{u_h(x_e) - u_h(x_\lambda)}{|x_\lambda - x_e|},$$

    where $x_\lambda$ is that point on the line through $x_f$ and $x_g$ (i.e.: $x_\lambda = \lambda\, x_f + (1-\lambda)\, x_g$), for which $x_\lambda - x_e$ is orthogonal to the edge $e$. Determine $\lambda$ and use the linearity of $u_h$ to express $u_h(x_\lambda)$ in terms of $u_h(x_f)$ and $u_h(x_g)$.

41. Assume the notations and assumptions from exercise 40.

    Let $V = H^1(\Omega)$ and $V_0 = H_0^1(\Omega)$, and let $V_h$ be the finite element space defined by the non-conforming Crouzeix-Raviart element (piecewise linear functions given by the values at the midpoints of the edges of $\mathcal{T}_h$), and let $V_{0h} \subset V_h$ be the sub-space with values 0 at the midpoint of edges on the boundary $\partial\Omega$.

    Let $f \in L_2(\Omega)$. Consider the variational formulation of the boundary value problem from exercise 38: Find $u_h \in V_{0h}$ such that

$$a_h(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in V_{0h} \tag{1}$$

with

$$a_h(u_h, v_h) = \sum_{r\in\mathbb{R}_h} \int_{\delta_r} \operatorname{grad} u_h(x) \cdot \operatorname{grad} v_h(x)\; dx, \quad \langle F, v_h\rangle = \int_\Omega f(x)\, v_h(x)\; dx.$$

Let $\phi_e \in V_h$ be the (nodal) basis function defined by

$$\phi_e(x_f) = \delta_{ef}. \quad \text{for all } f \in \mathcal{E}_h.$$

Show

$$a_h(u_h, \phi_e) = \left[\frac{\partial u_h}{\partial n}\right](x_e)\, |e| \quad \text{for all interior edges } e.$$

Hint: Observe that $\phi_e$ vanishes outside the two triangles $\delta_r$ and $\delta_s$ which share the common edge $e$. Use integration by parts on $\delta_r$ and $\delta_s$.

42. Assume the notations and assumptions from all previous exercises.

The finite volume method motivated by exercises 37, 38, 39, 40 is given by the conditions

$$\left[\frac{\partial u_h}{\partial n}\right](x_e)\, |e| = \int_{\mathcal{H}(x_e)} f(y)\; dy \quad \text{for all interior edges } e.$$

The finite element method from exercise 41 is given by the conditions

$$\left[\frac{\partial u_h}{\partial n}\right](x_e)\, |e| = \int_\Omega f(y)\, \phi_e(y)\; dy \quad \text{for all interior edges } e.$$

Show that the two methods are identical if applied to a piece-wise constant function $f(y)$, i.e.:

$$\int_{\mathcal{H}(x_e)} f(y)\; dy = \int_\Omega f(y)\, \phi_e(y)\; dy \quad \text{for all interior edges } e$$

for all $f \in \{v \in L_2(\Omega) : v|_{\delta_r} \in P_0 \text{ for all } r \in \mathbb{R}_h\}$.