**ÜBUNGEN ZU**

**NUMERIK ELLIPTISCHER PROBLEME**

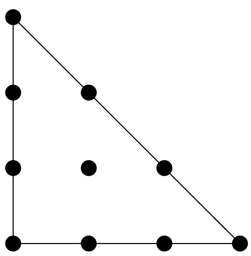
für 20.05.2008

---

31. Let $(\mathcal{T}_h)_{h\in\Theta}$ be a family of admissible subdivisions $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ of a bounded domain $\Omega \subset \mathbb{R}^2$ into triangles. The length of the longest edge of the triangle $\delta_r$ is denoted by $h^{(r)}$. Let $\Delta = \{\xi \in \mathbb{R}^2 : \xi_1 > 0, \xi_2 > 0, \xi_1 + \xi_2 < 1\}$ and let $x_{\delta_r} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be the usual bijective and affine linear mapping with $x_{\delta_r}(\Delta) = \delta_r$ for each $h \in \Theta$ and $r \in \mathbb{R}_h$. The Jacobian of $x_{\delta_r}$ is denoted by $J_{\delta_r}$.

    For each $m \in \mathbb{N}_0$, it can be shown that there exists a constant $c_1$ (depending only on $m$) such that
$$|v|_{H^m(\delta_r)} \leq c_1 \,|\det J_{\delta_r}|^{1/2}\, \|J_{\delta_r}^{-1}\|^m \,|v \circ x_{\delta_r}|_{H^m(\Delta)} \tag{1}$$
    and
$$|v \circ x_{\delta_r}|_{H^m(\Delta)} \leq c_1 \,|\det J_{\delta_r}|^{-1/2}\, \|J_{\delta_r}\|^m \,|v|_{H^m(\delta_r)} \tag{2}$$
    for all $h \in \Theta$, $r \in \mathbb{R}_h$ and all $v \in H^m(\delta_r)$. In class we showed (1) for $m = 1$ and (2) for $m = 2$.

    Consider the finite element space $V_h \subset H^1(\Omega)$, given by the shape functions $\mathcal{F}(\Delta) = P_k$ with $k \geq 1$ and the evaluations at all nodes $\xi^{(\alpha)} \in \{(\frac{i}{k}, \frac{j}{k}) : i, j \in \mathbb{N}_0 \text{ with } i+j \leq k\}$ as nodal variables $l^{(\alpha)}$, i.e., for example, for $k = 3$:



    In class we constructed a linear (interpolation) operator $I_h : H^2(\Omega) \longrightarrow V_h$ with
$$(I_h(v))(x_{\delta_r}(\xi)) = (\hat{I}(v \circ x_{\delta_r}))(\xi) \quad \text{for all } \xi \in \Delta \text{ and all } r \in \mathbb{R}_h,$$
    for the corresponding linear (interpolation) operator $\hat{I} : H^2(\Delta) \longrightarrow P_k$ on the reference element. For all integers $s$ and $l$ with $0 \leq s \leq l$ and $2 \leq l \leq k+1$ it can be shown that there exists a constant $c_2$ (depending only on $s$ and $l$) with
$$|\hat{v} - \hat{I}(\hat{v})|_{H^s(\Delta)} \leq c_2 \,|\hat{v}|_{H^l(\Delta)} \quad \text{for all } \hat{v} \in H^l(\Delta). \tag{3}$$
    In class we showed (3) for $s = 1$ and $l = 2$.

    For $m \in \mathbb{N}_0$ consider the so-called broken Sobolev space $H^m(\Omega, \mathcal{T}_h)$, given by
$$H^m(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega) : v|_{\delta_r} \in H^m(\delta_r) \text{ for all } r \in \mathbb{R}_h\}$$

with semi-norm

$$|v|_{H^m(\Omega,\mathcal{T}_h)} = \left( \sum_{r\in\mathbb{R}_h} |v|^2_{H^m(\delta_r)} \right)^{1/2} .$$

It is obvious that

$$H^m(\Omega) \subset H^m(\Omega, \mathcal{T}_h) \quad \text{and} \quad |v|_{H^m(\Omega)} = |v|_{H^m(\Omega,\mathcal{T}_h)} \quad \text{for all } v \in H^m(\Omega).$$

Assume that there are constant $c_4$, $c_5$ such that

$$\|J_{\delta_r}\| \le c_4\, h^{(r)} \quad \text{for all } h \in \Theta,\ r \in \mathbb{R}_h, \tag{4}$$

$$\|J_{\delta_r}^{-1}\| \le c_5/h^{(r)} \quad \text{for all } h \in \Theta,\ r \in \mathbb{R}_h. \tag{5}$$

Use the conditions (1) - (5) to show that, for all integers $s$ and $l$ with $0 \le s \le l$ and $2 \le l \le k+1$, there exists a constant $c_6$ (depending only on $c_1$, $c_2$, $c_3$, $c_4$ and $c_5$) such that

$$|v - I_h(v)|_{H^s(\Omega,\mathcal{T}_h)} \le c_6\, h^{l-s}\, |v|_{H^l(\Omega)} \quad \text{for all } v \in H^l(\Omega)$$

with $h = \max_{r\in\mathbb{R}_h} h^{(r)}$.

32. Assume the notations and assumptions of exercise 31 except that (3) is replaced by weaker condition: There exists a constant $c_2$ (depending only on $s$ and $l$) with

$$|\hat{v} - \hat{I}(\hat{v})|_{H^s(\Delta)} \le c_2\, \|\hat{v}\|_{H^l(\Delta)} \quad \text{for all } \hat{v} \in H^l(\Delta). \tag{6}$$

What kind of estimate for $v - I_h(v)$ can be derived under these circumstances?

33. Assume the notations and assumptions of exercise 31. Show:

(a) For all integers $s$ and $l$ with $0 \le s \le 1$ and $2 \le l \le k+1$, there exists a constant $c_7$ such that

$$|v - I_h(v)|_{H^s(\Omega)} \le c_7\, h^{l-s}\, |v|_{H^l(\Omega)} \quad \text{for all } v \in H^l(\Omega).$$

(b) For all integers $s$ and $l$ with $0 \le s \le l$ and $2 \le l \le k+1$ there exists a constant $c_8$ such that

$$|I_h(v)|_{H^s(\Omega,\mathcal{T}_h)} \le c_8\, \|v\|_{H^l(\Omega)} \quad \text{for all } v \in H^l(\Omega).$$

Hint: $I_h(v) = I_h(v) - v + v$.

(c) There exists a constant $c_9$ such that

$$\|v - I_h(v)\|_{L^2(\Omega)} + h\, |v - I_h(v)|_{H^1(\Omega)} \le c_9\, h^2\, |v|_{H^2(\Omega)} \quad \text{for all } v \in H^2(\Omega).$$

34. Assume the notations and assumptions of exercise 31 for the case $k = 0$, $s = 0$ and $l = 1$ with the following modifications:

$V_h \subset L^2(\Omega)$ is the set of piecewise constant functions with respect to the subdivision. $I_h : L^2(\Omega) \longrightarrow V_h$ is now given (element-wise) by

$$I_h(v)(x) = \overline{v}_{\delta_r} \quad \text{for all } x \in \delta_r,$$

where $\overline{v}_{\delta_r}$ denotes the mean value of $v$ on $\delta_r$:

$$\overline{v}_{\delta_r} = \frac{1}{\operatorname{meas}(\delta_r)} \int_{\delta_r} v \, dx.$$

Show:

(a) Show that

$$(I_h(v))(x_{\delta_r}(\xi)) = (\hat{I}(v \circ x_{\delta_r}))(\xi) \quad \text{for all } \xi \in \Delta \text{ and all } r \in \mathbb{R}_h,$$

for an appropriate linear operator $\hat{I} : L^2(\Delta) \longrightarrow P_0$ on the reference element.

(b) Show (3) for the new operator $\hat{I}$.
Hint: Apply Poincaré's inequality for the function $\hat{v} - \hat{I}(\hat{v})$.

(c) Show that there exists a constant $c_{10}$ such that

$$\|v - I_h(v)\|_{L^2(\Omega)} \le c_{10} \, h \, |v|_{H^1(\Omega)} \quad \text{for all } v \in H^1(\Omega).$$

35. Assume the notations and assumptions of exercise 31. Additionally assume that the family of triangulations is quasi-uniform. Show that there exists a constant $c_{11}$ such that the inverse inequality

$$\|v_h\|_{H^1(\Omega)} \le c_{11} \frac{1}{h} \|v_h\|_{L^2(\Omega)} \quad \text{for all } v_h \in V_h$$

is satisfied.

36. Assume the notations and assumptions of exercise 35. Does a constant $c_{12}$ exist such that the inverse of the inverse inequality

$$\frac{1}{h} \|v_h\|_{L^2(\Omega)} \le c_{12} \|v_h\|_{H^1(\Omega)} \quad \text{for all } v_h \in V_h$$

is satisfied?