ÜBUNGEN ZU

NUMERIK ELLIPTISCHER PROBLEME

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25. Assume the notations and the assumptions of exercise 19.

    Show that the family of subdivisions $(\mathcal{T}_h)_{h\in\Theta}$ is shape-regular if and only if there exist positive constants $\underline{c}_1, \overline{c}_1, c_2, c_3$ such that

    (a) $\underline{c}_1\,(h^{(r)})^2 \le |\det J_{\delta_r}| \le \overline{c}_1\,(h^{(r)})^2$ for all $h\in\Theta$, $r\in\mathbb{R}_h$,

    (b) $\|J_{\delta_r}\|_{\ell^2} \le c_2\,h^{(r)}$ for all $h\in\Theta$, $r\in\mathbb{R}_h$, and

    (c) $\|J_{\delta_r}^{-1}\|_{\ell^2} \le c_2\,(h^{(r)})^{-1}$ for all $h\in\Theta$, $r\in\mathbb{R}_h$.

26. Assume the notations and the assumptions of exercise 19 and let the family of subdivisions $(\mathcal{T}_h)_{h\in\Theta}$ be shape-regular.

    V_h denotes the corresponding finite element space of the Courant element.

    Show: There exist positive constants $\underline{c}_1, \overline{c}_1$ such that

$$\underline{c}_1\left[\min_{r\in\mathbb{R}_r} h^{(r)}\right]^2 (\underline{v}_h,\underline{v}_h)_{\ell^2} \le (v_h,v_h)_{H^1(\Omega)} \le \overline{c}_1\,(\underline{v}_h,\underline{v}_h)_{\ell^2} \quad \text{for all } v_h\in V_h.$$

27. Assume the notations and the assumptions of exercise 19 and let the family of subdivisions $(\mathcal{T}_h)_{h\in\Theta}$ be shape-regular.

    V_h denotes the corresponding finite element space of the Courant element.

    Show: There exist positive constants $\underline{c}_0, \overline{c}_0$ such that

$$\underline{c}_0\left[\min_{r\in\mathbb{R}_r} h^{(r)}\right]^2 (\underline{v}_h,\underline{v}_h)_{\ell^2} \le (v_h,v_h)_{L^2(\Omega)} \le \overline{c}_0\left[\max_{r\in\mathbb{R}_r} h^{(r)}\right]^2 (\underline{v}_h,\underline{v}_h)_{\ell^2} \quad \text{for all } v_h\in V_h.$$

28. Use the estimates from the exercises 26 and 27 to derive estimates

$$\underline{\gamma}\,(v_h,v_h)_{L^2(\Omega)} \le (v_h,v_h)_{H^1(\Omega)} \le \overline{\gamma}\,(v_h,v_h)_{L^2(\Omega)} \quad \text{for all } v_h\in V_h,$$

    with appropriate (not necessarily constant with respect to $(\mathcal{T}_h)_{h\in\Theta}$) positive coefficients $\underline{\gamma}$ and $\overline{\gamma}$, which are independent of $v_h\in V_h$.

29. Assume the notations and assumptions of exercise 19 and let V_h be the corresponding finite element space of the Courant element.

    The 3×3-matrices G_0 and G_1 are given by

$$G_0 = ((p^{(\alpha)}, p^{(\beta)})_{L^2(\Delta)})_{\alpha,\beta\in A}, \quad G_1 = ((p^{(\alpha)}, p^{(\beta)})_{H^1(\Delta)})_{\alpha,\beta\in A},$$

where $\{p^{(\alpha)} : \alpha \in A = \{1,2,3\}\}$ denotes the nodal basis on the reference element.

Show: There exist positive constants $\underline{c}, \overline{c}$ such that

$$\underline{c}\,(G_0\underline{v},\underline{v})_{\ell^2} \leq (G_1\underline{v},\underline{v})_{\ell^2} \leq \overline{c}\,(G_0\underline{v},\underline{v})_{\ell^2} \quad \text{for all } \underline{v} \in \mathbb{R}^3.$$

Hint: You can use (without proof) that

$$\lambda_{\min}(G)\,(\underline{v},\underline{v})_{\ell^2} \leq (G\underline{v},\underline{v})_{\ell^2} \leq \lambda_{\max}(G)\,(\underline{v},\underline{v})_{\ell^2} \quad \text{for all } \underline{v} \in \mathbb{R}^3.$$

for all symmetric matrices G.

30. Assume the notations and assumptions of exercise 29 and let the family of subdivisions $(\mathcal{T}_h)_{h\in\Theta}$ be shape-regular.

Show: There exist positive constants $\underline{c}, \overline{c}$ such that

$$\underline{c}\,(v_h, v_h)_{L^2(\Omega)} \leq (v_h, v_h)_{H^1(\Omega)} \leq \overline{c}\,\left[\min_{r\in\mathbb{R}_r} h^{(r)}\right]^{-2} (v_h, v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in V_h.$$

Hint: Use the estimates from exercise 29 for the upper bound.