## ÜBUNGEN ZU

## NUMERIK ELLIPTISCHER PROBLEME

für 29.04.2008

---

19. Let $(\mathcal{T}_h)_{h\in\Theta}$ be a family of subdivisions $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ of a domain $\Omega \subset \mathbb{R}^2$ into non-degenerate triangles. For a triangle $\delta_r \in \mathcal{T}_h$, its vertices are denoted by $x^{(r,\alpha)}$ with corresponding angles $\theta_\alpha^{(r)}$, $\alpha \in A = \{1,2,3\}$, the length of the edge opposite to $x_\alpha^{(r)}$ is denoted by $h_\alpha^{(r)}$, $\alpha \in A = \{1,2,3\}$, its diameter is given by $h^{(r)} = \max\{h_\alpha^{(r)} : \alpha \in A\}$, and the radius of its largest inscribed circle is denoted by $\rho^{(r)}$. Furthermore, let $\Delta = \{\xi \in \mathbb{R}^2 : \xi_1 > 0, \xi_2 > 0, \xi_1 + \xi_2 < 1\}$ be the reference triangle. The mapping $x_{\delta_r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which maps $\Delta$ to $\delta_r$ is given by

$$x_{\delta_r}(\xi) = x^{(r,1)} + J_{\delta_r}\,\xi \quad \text{with } J_{\delta_r} = (x^{(r,2)} - x^{(r,1)}, x^{(r,3)} - x^{(r,1)}) \in \mathbb{R}^{2\times 2}.$$

Show that the following two statements are equivalent:

(a) There exists a constant $\theta_0 > 0$ such that $\theta_0 \le \theta_\alpha^{(r)}$ for all $h \in \Theta$, $r \in \mathbb{R}_h$, $\alpha \in A$.

(a') There exists a constant $\sigma_0 > 0$ such that $\sigma_0\, h^{(r)} \le \rho^{(r)}$ for all $h \in \Theta$, $r \in \mathbb{R}_h$.

Hint: Show and use

$$\rho^{(r)} \left[\cot\left(\frac{\theta_\beta^{(r)}}{2}\right) + \cot\left(\frac{\theta_\gamma^{(r)}}{2}\right)\right] = h_\alpha^{(r)},$$

for pairwise dinstinct indices $\alpha, \beta, \gamma \in A$.

20. Assume the notations and assumptions of exercise 19.

Show that there exist constants $\underline{c}_1, \overline{c}_1 > 0$ such that

$$\underline{c}_1\, \rho^{(r)}\, h^{(r)} \le |\det J_{\delta_r}| \le \overline{c}_1\, \rho^{(r)}\, h^{(r)} \quad \text{for all } h \in \Theta, r \in \mathbb{R}_h.$$

Hint: Show and use

$$2\,\mathrm{meas}(\delta_r) = (h_1^{(r)} + h_2^{(r)} + h_3^{(r)})\,\rho^{(r)}.$$

21. Assume the notations and assumptions of exercise 19.

Show that there exist constants $c_2, c_3 > 0$ such that

$$\|J_{\delta_r}\|_{\ell^2} \le c_2\, h^{(r)} \quad \text{and} \quad \|J_{\delta_r}^{-1}\|_{\ell^2} \le c_3\,\frac{1}{\rho^{(r)}} \quad \text{for all } h \in \Theta, r \in \mathbb{R}_h.$$

Here, $\|M\|_{\ell^2}$ denotes the spectral norm of a matrix $M$:

$$\|M\|_{\ell^2} = \sup_{v\neq 0} \frac{\|Mv\|_{\ell^2}}{\|v\|_{\ell^2}}$$

with the Euclidean vector norm $\|.\|_{\ell^2}$.

Hint: Show and use

$$\|J_{\delta_r}\|_{\ell^2} = \sup_{\xi \in B_R(\eta)} \frac{\|x_{\delta_r}(\xi) - x_{\delta_r}(\eta)\|_{\ell^2}}{\|\xi - \eta\|_{\ell^2}}$$

with $B_R(\eta) = \{\xi \in \mathbb{R}^2 : \|\xi - \eta\| = R\}$ for all $\eta \in \mathbb{R}^2$ and for all $R > 0$. In particular, consider the largest inscribed circle of $\Delta$ for $B_R(\eta)$, in order to show the first upper bound.

22. Show: A family of triangulations is regular if and only if it is quasi-uniform.

23. Let $(\mathcal{T}_h)_{h\in\Theta}$ be a family of subdivisions $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ of a domain $\Omega \subset \mathbb{R}^2$ into non-degenerate convex quadrilaterals. Let $\Delta = (0,1) \times (0,1)$ be the unit square with vertices $\xi^{(1)} = (0,0)$, $\xi^{(2)} = (1,0)$, $\xi^{(3)} = (1,1)$ and $\xi^{(4)} = (0,1)$. For a quadrilateral $\delta_r \in \mathcal{T}_h$, its vertices are denoted by $x^{(r,\alpha)}$, $\alpha \in A = \{1,2,3,4\}$ (numbered counterclockwise). Let $x_{\delta_r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the unique mapping with $x_{\delta_r} \in Q_1$ and $x_{\delta_r}(\xi^{(\alpha)}) = x^{(r,\alpha)}$. The Jacobian of $x_\delta$ at a point $\xi \in \mathbb{R}^2$ is denoted by $J_{\delta_r}(\xi)$.

    Show:

    (a) $\det J_{\delta_r}(\xi) \in P_1$.

    (b) $\det J_{\delta_r}(\xi^{(\alpha)}) = 2 \operatorname{meas}(S_\alpha^{(r)})$ for all $\alpha \in A$, where $S_\alpha^{(r)}$ denotes the triangle whose vertices are $x^{(r,\alpha)}$ and the two vertices of $\delta_r$ which are connected to $x^{(r,\alpha)}$ by an edge of $\delta_r$.

    (c) $\det J_{\delta_r}(\xi) \geq 0$ for all $\xi \in \overline{\Delta}$.

    Hint: Show and use

$$x_{\delta_r}(\xi) = \sum_{\alpha=1}^{4} p^{(\alpha)}(\xi)\, x^{(r,\alpha)}.$$

24. Assume the notations and assumptions of exercise 23.

    Show that there exist constants $\underline{c}_1, \overline{c}_1$ such that

$$\underline{c}_1\, (\rho^{(r)})^2 \leq |\det J_{\delta_r}(\xi)| \leq \overline{c}_1\, (h^{(r)})^2 \quad \text{for all } h \in \Theta, r \in \mathbb{R}_h$$

    with $h^{(r)} = \max\{\|y - x\| : x, y \in \overline{\delta_r}\}$ (the diameter of $\delta_r$) and $\rho^{(r)} = 2 \min\{\rho_\alpha^{(r)} : \alpha \in A\}$, where $\rho_\alpha^{(r)}$ denotes the radius of the largest inscribed circle of $S_\alpha^{(r)}$.

    Hint: The functional $\det J_{\delta_r}(\xi)$ attains its minimum (maximum) at some vertex of $\Delta$.