ÜBUNGEN ZU

NUMERIK ELLIPTISCHER PROBLEME

für 22.04.2008

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13. (a) Consider the isoparametric bilinear element on admissible quadrilateral subdivisions, given by the reference domain $\Delta = [-1,1]^2$, the shape functions

$$\mathcal{F}(\Delta) = \operatorname{span}(1, \xi_1, \xi_2, \xi_1\xi_2) = Q_1$$

and the nodal variables defined by the function evaluations at the four vertices of Δ. Let δ denote the element domain which is obtained by rotating Δ by 45° counterclockwise around the origin. The corresponding (affine linear) transformation is denoted by x_δ. Show for the transformed shape functions $\mathcal{F}(\delta) = \{v = \hat{v} \circ x_\delta^{-1} : \hat{v} \in \mathcal{F}(\Delta)\}$:

$$\mathcal{F}(\delta) = \operatorname{span}(1, x_1, x_2, x_1^2 - x_2^2) \neq Q_1.$$

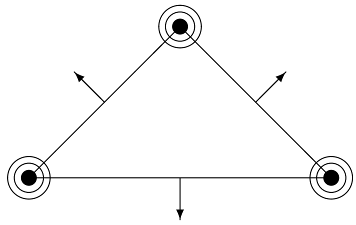
(b) As an alternative, consider the set Q_1 as shape functions on δ. Is such a shape function uniquely determined by its values at the four vertices of δ?

14. Let $\delta \subset \mathbb{R}^2$ be a general non-degenerate quadrilateral obtained from the reference element $\Delta = [-1,1]^2$ by a bilinear transformation x_δ which maps the vertices of Δ to the vertices of δ.

The so-called rotated bilinear finite element on δ is given by the shape functions $\mathcal{F}(\delta) = \{v = \hat{v} \circ x_\delta^{-1} : \hat{v} \in \mathcal{F}(\Delta)\}$ with $\mathcal{F}(\Delta) = \operatorname{span}(1, \xi_1, \xi_2, \xi_1^2 - \xi_2^2)$ and the nodal variables defined by the function evaluations at the midpoints of the four edges of δ. Show that these nodal variables uniquely determine a shape function.

15. Assume the notations and assumptions of exercise 14. Show by a simple example that this finite element is not a C^0-element on admissible quadrilateral subdivisions.

16. Show that the Argyris element



is a C^1-element on admissible triangulations.

Hint: Show that the values of a shape function and its normal derivative along an edge are uniquely determined by the prescribed values on this edge.

17. Consider the Raviart-Thomas element, given by

(i) a non-degenerate tetrahedron $\delta \subset \mathbb{R}^3$, whose midpoints and outer normal unit vectors of the four faces are denoted by $x^{(\alpha)}$ and $n^{(\alpha)}$, respectively,

(ii) the shape functions

$$\mathcal{F}(\delta) = \{v(x) = a + b\, x : a \in \mathbb{R}^3,\, b \in \mathbb{R}\}$$

and

(iii) the nodal variables

$$l^{(\alpha)}(v) = v(x^{(\alpha)}) \cdot n^{(\alpha)}.$$

Show property (b) of exercise 7.

Hint: Show first: If $l^{(\alpha)}(v) = 0$, then $v(x) \cdot n^{(\alpha)} = 0$ on the whole face containing $x^{(\alpha)}$. Observe that each vertex of the tetrahedron belongs to three different faces.

18. Consider the Nedelec element, given by

(i) a non-degenerate tetrahedron $\delta \subset \mathbb{R}^3$, whose unit vectors along the edges and midpoints of the edges are denoted by $t^{(\alpha)}$ and $x^{(\alpha)}$, respectively,

(ii) the shape functions

$$\mathcal{F}(\delta) = \{v(x) = a + b \times x : a \in \mathbb{R}^3,\, b \in \mathbb{R}^3\}$$

and

(iii) the nodal variables

$$l^{(\alpha)}(v) = v(x^{(\alpha)}) \cdot t^{(\alpha)}.$$

Show property (b) of exercise 7.

Hint: Show first: If $l^{(\alpha)}(v) = 0$, then $v(x) \cdot t^{(\alpha)} = 0$ on the whole edge containing $x^{(\alpha)}$. Observe that each vertex of the tetrahedron belongs to three different edges.