ÜBUNGEN ZU

NUMERIK ELLIPTISCHER PROBLEME

für 15.04.2008

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7. Let

   (i) $\delta \subset \mathbb{R}^d$ be a domain with piecewise smooth boundary (the element domain),

   (ii) $\mathcal{F}(\delta)$ be a finite-dimensional space of functions on δ (the shape functions) and

   (iii) $\{l^{(\alpha)} : \alpha \in A\}$ be a finite set of linear functionals on $\mathcal{F}(\delta)$ (the nodal variables).

   Let $\mathcal{F}(\delta)$ be an $|A|$-dimensional vector space, where $|A|$ denotes the cardinality of the index set A. Show that the following three statements are equivalent:

   (a) $\{l^{(\alpha)} : \alpha \in A\}$ is a basis of $\mathcal{F}(\delta)^*$ (the dual space of $\mathcal{F}(\delta)$).

   (b) For all $v \in \mathcal{F}(\delta)$: If $l^{(\alpha)}(v) = 0$ for all $\alpha \in A$, then $v(x) = 0$ for all $x \in \delta$.

   (c) For all values $v^{(\alpha)} \in \mathbb{R}$, $\alpha \in A$, there exists a unique function $v \in \mathcal{F}(\delta)$ with $l^{(\alpha)}(v) = v^{(\alpha)}$ for all $\alpha \in A$.

   Hint: Let $\{p^{(\alpha)} : \alpha \in A\}$ be a basis of $\mathcal{F}(\delta)$. Each of the three statements corresponds directly to a property of the square matrix $M = \left(l^{(\alpha)}(p^{(\beta)})\right)_{\alpha,\beta \in A}$.

8. Let $L : \mathbb{R}^d \longrightarrow \mathbb{R}$ be given by
$$L(x) = a \cdot x + b$$
   with $0 \neq a \in \mathbb{R}^d$, $b \in \mathbb{R}$, where $\cdot$ denotes the Euclidean inner product. The set $\ker L = \{x \in \mathbb{R}^d : L(x) = 0\}$ is called a hyperplane. Let $p \in P_k$ (the set of polynomials in x of degree $\leq k$).

   Show: If p vanishes on the hyperplane $\ker L$, then we can write $p = L\,q$, where $q \in P_{k-1}$.

   Hint: Show the statement first for the special hyperplane, given by $x_d = 0$. Then use the fact that there is an affine change of coordinates $\hat{x} = T(x)$ with $\hat{x}_d = L(x)$, which transforms the hyperplane $\ker L$ to the hyperplane, given by $x_d = 0$.

9. Consider the Courant finite element, given by

   (i) a non-degenerate triangle δ as element domain, whose vertices are denoted by $x^{(\alpha)}$ for $\alpha \in A = \{1, 2, 3\}$,

   (ii) the shape functions $\mathcal{F}(\delta) = P_1$ and

   (iii) the nodal variables $l^{(\alpha)}$, given by
$$l^{(\alpha)}(v) = v(x^{(\alpha)}) \quad \text{for } \alpha \in A.$$

Use exercise 8 to show property (b) of exercise 7.

Hint: Use the hyperplane (straight line) determined by an edge of the triangle to show that $v = L\,q$ with $q \in P_0$.

10. Use exercise 8 to show property (b) of exercise 7 for the quadratic triangular Lagrange finite element, given by

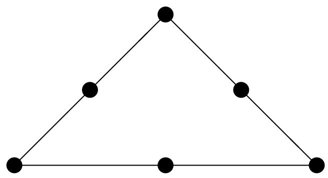
   (i) a non-degenerate triangle δ as element domain, whose vertices are denoted by $x^{(\alpha)}$ for $\alpha \in \{1,2,3\}$ and whose midpoints of the edges are denoted by $x^{(\alpha)}$ for $\alpha \in \{4,5,6\}$,

   (ii) the shape functions $\mathcal{F}(\delta) = P_2$ and

   (iii) the nodal variables $l^{(\alpha)}$ for $\alpha \in A = \{1,2,3,4,5,6\}$ are given by:

$$l^{(\alpha)}(v) = v(x^{(\alpha)})$$

   for $\alpha \in A$.



Hint: Use the same technique as in exercise 9 consecutively for two edges.

11. Use exercise 8 to show property (b) of exercise 7 for the cubic Hermite finite element, given by

   (i) a non-degenerate triangle δ as element domain, whose vertices are denoted by $x^{(\alpha)}$ for $\alpha \in \{1,2,3\}$ and whose centroid is denoted by $x^{(4)}$,

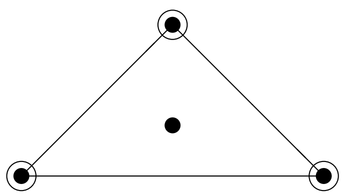
   (ii) the shape functions $\mathcal{F}(\delta) = P_3$ and

   (iii) the nodal variables $l^{(\alpha)}$ for $\alpha \in A = \{1,2,\ldots,10\}$ are given by:

$$l^{(\alpha)}(v) = v(x^{(\alpha)}) \quad \text{for } \alpha \in \{1,2,3,4\}$$

   and

$$l^{(2\alpha+3)}(v) = \frac{\partial v}{\partial x_1}(x^{(\alpha)}), \quad l^{(2\alpha+4)}(v) = \frac{\partial v}{\partial x_2}(x^{(\alpha)}) \quad \text{for } \alpha \in \{1,2,3,\}.$$



Hint: Use the same technique as in exercise 9 consecutively for the three edges.

12. Let $\delta = \Delta = \{\xi \in \mathbb{R}^2 : \xi_1 > 0, \xi_2 > 0, \xi_1 + \xi_2 < 1\}$. Find an explicit formula in ξ for the nodal basis function $p^{(1)}(\xi)$ (associated to the node $\xi^{(1)} = (0, 0)$) for the finite element of exercise 11.