ÜBUNGEN ZU

NUMERIK ELLIPTISCHER PROBLEME

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1. Consider the following boundary value problem: Find u such that

$$-\operatorname{div}(A(x)\operatorname{grad} u(x)) + b(x)\cdot \operatorname{grad} u(x) + c(x)\,u(x) = f(x) \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial N}(x) = 0 \quad \text{on } \partial\Omega$$

with given data $A \in L_\infty(\Omega)^{d\times d}$, $b \in L_\infty(\Omega)^d$, $c \in L_\infty(\Omega)$, $f \in L_2(\Omega)$ under the following assumptions: There exist constants $\underline{a} > 0$, $\overline{a} > 0$, $\overline{b} \geq 0$, $\underline{c} > 0$, and $\overline{c} > 0$ with

   (a) $A(x)$ is symmetric and $\underline{a}\,|\xi|^2 \leq A(x)\xi\cdot\xi \leq \overline{a}\,|\xi|^2$ for all $x \in \Omega, \xi \in \mathbb{R}^d$,
   (b) $|b(x)| \leq \overline{b}$ for all $x \in \Omega$,
   (c) $\underline{c} \leq c(x) \leq \overline{c}$ for all $x \in \Omega$,
   (d) $\overline{b}^2 < 4\,\underline{a}\,\underline{c}$.

The corresponding variational formulation leads to a problem of the form: Find $u \in H^1(\Omega)$ such that

$$a(u,v) = \langle F, v\rangle \quad \text{for all } v \in H^1(\Omega).$$

Determine $a(u,v)$, $\langle F, v\rangle$, and constants μ_2, C such that

$$|a(u,v)| \leq \mu_2\,\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \quad \text{for all } u,v \in H^1(\Omega)$$

and

$$|\langle F, v\rangle| \leq C\,\|v\|_{H^1(\Omega)} \quad \text{for all } v \in H^1(\Omega).$$

2. Assume the notations and assumptions of Example 1. Determine a constant $\mu_1 > 0$ such that

$$a(v,v) \geq \mu_1\,\|v\|^2_{H^1(\Omega)} \quad \text{for all } v \in H^1(\Omega).$$

Hint: Show first

$$a(v,v) \geq q(\|v\|_{L_2(0,1)}, |v|_{H^1(0,1)})$$

with $q(\xi_0,\xi_1) = \underline{a}\,\xi_1^2 - \overline{b}\,\xi_1\xi_0 + \underline{c}\,\xi_0^2$. Then show and use: $q(\xi_0,\xi_1) \geq \underline{a}\,C\,\xi_1^2$ and $q(\xi_0,\xi_1) \geq \underline{c}\,C\,\xi_0^2$ with $C = 1 - \overline{b}^2/(4\underline{a}\,\underline{c})$.

3. Let $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}$, $\psi : \overline{\Omega} \longrightarrow \mathbb{R}$, $u : \overline{\Omega} \longrightarrow \mathbb{R}^3$, $v : \overline{\Omega} \longrightarrow \mathbb{R}^3$ be sufficiently smooth functions on a bounded domain $\overline{\Omega} = \Omega \cup \partial\Omega$ with a suffiently smooth boundary $\partial\Omega$. Complete the following identities by integration by parts:

$$\int_\Omega (\operatorname{grad} \varphi(x))\, \psi(x)\; dx = -\int_\Omega \varphi(x) \ldots\; dx + \int_{\partial\Omega} \varphi(x) \ldots\; ds$$

$$\int_\Omega (\operatorname{grad} \varphi(x)) \cdot v(x)\; dx = -\int_\Omega \varphi(x) \ldots\; dx + \int_{\partial\Omega} \varphi(x) \ldots\; ds$$

$$\int_\Omega (\operatorname{div} u(x))\, \psi(x)\; dx = -\int_\Omega u(x) \ldots\; dx + \int_{\partial\Omega} u(x) \ldots\; ds$$

$$\int_\Omega (\operatorname{div} u(x))\, v(x)\; dx = -\int_\Omega \ldots u(x) \ldots\; dx + \int_{\partial\Omega} u(x) \ldots\; ds$$

$$\int_\Omega (\operatorname{curl} u(x))\, \psi(x)\; dx = \int_\Omega u(x) \ldots\; dx - \int_{\partial\Omega} u(x) \ldots\; ds$$

$$\int_\Omega (\operatorname{curl} u(x)) \cdot v(x)\; dx = \int_\Omega u(x) \ldots\; dx - \int_{\partial\Omega} u(x) \ldots\; ds$$

4. Show that the equilibrium condition

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega,$$

Hooke's law

$$\sigma = \lambda \operatorname{div} u\, I + 2\mu\, \varepsilon(u)$$

with constant positive parameters λ, μ and

$$\varepsilon(u) = \frac{1}{2} \left[\operatorname{grad} u + (\operatorname{grad} u)^T \right]$$

lead to the Lamé equations

$$-\mu\, \Delta u(x) - (\lambda + \mu) \operatorname{grad}(\operatorname{div} u(x)) = f(x) \quad \text{in } \Omega.$$

5. Multiply the Lamé equations by a test function $v(x)$ and integrate over the domain Ω. Then eliminate second-order derivatives by integration by parts and put the result into the following form

$$\int_\Omega \ldots\; dx - \int_{\partial\Omega} \ldots\; ds = \int_\Omega f \cdot v\; dx.$$

Check whether the first integral (over the domain Ω) and the second integral (over the boundary $\partial\Omega$) conindice with the bilinear form

$$a(u, v) = \int_\Omega \sigma : \varepsilon(v)\; dx = \int_\Omega \left[\lambda \operatorname{div} u \operatorname{div} v + 2\mu\, \varepsilon(u) : \varepsilon(v)\right]\; dx$$

and the expression

$$\int_{\partial\Omega} \sigma n \cdot v \, ds = \int_{\partial\Omega} \left[\lambda \,(\operatorname{div} u)\,(v \cdot n) + 2\mu\, \varepsilon(u) n \cdot v\right] \, ds,$$

respectively.

6. Let $\Omega = \{x \in \mathbb{R}^3 : x_1 > 0,\ x_2 > 0,\ x_3 > 0, x_1^2 + x_2^2 + x_3^2 < 1\}$ and consider the functions

$$\varphi(x) = r^\alpha \quad \text{with } r = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad \text{and} \quad v = \operatorname{grad} \varphi.$$

Show that

$$\operatorname{curl} v = 0 \quad \text{on } \Omega.$$

Find an exponenent $\alpha \in \mathbb{R}$ such that

$$\int_\Omega |v(x)|^2 \, dx < \infty \quad \text{and} \quad \int_\Omega |\operatorname{div} v|^2 \, dx = \infty,$$

(which implies $v \in H(\text{curl}, \Omega)$ but $v \notin H^1(\Omega)^3$).

Hint: Use spherical coordinates.