ÜBUNGEN ZU NUMERIK ZEITABHÄNGIGER PROBLEME

für den 28.1.2008

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60. Show that the initial value problem

$$\begin{aligned} Bu'(t) + Au(t) &= f(t), \\ u(0) &= u_0 \end{aligned}$$

has at most one solution if $A + \lambda B$ is a regular matrix stencil.

Hint: Consider the homogeneous problem.

61. Show that the initial value problem

$$\begin{aligned} Bu'(t) + Au(t) &= 0, \\ u(0) &= 0 \end{aligned}$$

has infinitely many different solutions if $A + \lambda B$ is not a regular matrix stencil.

Hint: Construct solutions of the form $u(t) = \sum_i \alpha_i e^{\lambda_i t} v_i$ with distinct values λ_i and vectors $v_i \neq 0$ satisfying $(A + \lambda_i B)v_i = 0$.

62. Let $N \in \mathbb{R}^{n\times n}$ and $\nu \in \mathbb{N}$ with $N^\nu = 0$. Show:

$$(I - N)^{-1} = \sum_{i=0}^{\nu-1} N^i.$$

63. Consider the following formulation of a constrained mechanical system:

$$\begin{aligned} q' &= u \\ M(q)u' &= f(q,u) - G(q)^T\lambda \\ 0 &= g(q) \end{aligned} \tag{1}$$

with a symmetric positive definite matrix $M(q) \in \mathbb{R}^{n\times n}$, $G(q) = (\partial g/\partial q)(q) \in \mathbb{R}^{m\times n}$ with full rank $m \leq n$. Show that this system is equivalent to the GGL formulation:

$$\begin{aligned} q' &= u - G(q)^T\mu \\ M(q)u' &= f(q,u) - G(q)^T\lambda \\ 0 &= g(q) \\ 0 &= G(q)u. \end{aligned}$$

That means: If $q(t)$, $u(t)$, $\lambda(t)$ satisfy (1), then $q(t)$, $u(t)$, $\lambda(t)$, and $\mu(t) = 0$ satisfy the GGL formulation. If $q(t)$, $u(t)$, $\lambda(t)$, and $\mu(t)$ satisfy the GGL formulation, then $q(t)$, $u(t)$, $\lambda(t)$ satisfy (1).

Hint for the second part: Multiply the first equation by $G(q)$, compare with the differentiated third equation and the fourth equationm and conclude $\mu(t) = 0$.

64. Rewrite the GGL formulation as a Hessenberg index 2 system:

$$\begin{aligned} x_1'(t) &= f_1(t, x_1(t), x_2(t)), \\ 0 &= f_2(t, x_1(t)) \end{aligned}$$

with a nonsingular matrix

$$\frac{\partial f_2}{\partial x_1}(t, x_1)\frac{\partial f_1}{\partial x_2}(t, x_1, x_2).$$

65. Consider the following optimal control problem for the unknowns y, p and u:

$$\begin{aligned} y'(t) &= Ay(t) + Bu(t) + f(t), \quad y(0) = y_0, \\ p'(t) &= -A^T p(t) - Cy(t), \quad p(1) = 0, \\ 0 &= B^T p(t) + Du(t) \end{aligned}$$

with positive semi-definite matrices C and D. Show:

(a) If D is positive definite, then the optimal control problem has index 1.

(b) If $D = 0$ and $B^T CB$ is positive definite, then the optimal control problem has index 3.