# ÜBUNGEN ZU NUMERIK ZEITABHÄNGIGER PROBLEME

für den 18.1.2008

---

54. Determine all parameters $\theta \in \mathbb{R}$ such that the $\theta$-method

$$u_{j+1} = u_j + \tau\ [(1-\theta)\, f(t_j, u_j) + \theta\, f(t_{j+1}, u_{j+1})]$$

is algebraically stable.

55. The inner product $A : B$ of two matrices $A, B \in \mathbb{R}^{n\times n}$ is given by

$$A : B = \sum_{i,j=1}^{n} a_{ij} b_{ij}.$$

Show: If $A$ and $B$ are symmetric and positiv semi-definit, then $A : B \geq 0$.

Hint: Let $a_j, b_j \in \mathbb{R}^n$ denote the $j$-th column vectors of $A$ and $B$, respectively. Then

$$A : B = \sum_{j=1}^{n} a_j \cdot b_j,$$

where the symbol $\cdot$ denotes the Euclidean inner product of vectors. Show first:

$$(QA) : (QB) = A : B = (AQ^T) : (BQ^T) \quad \text{for all orthogonal matrices } Q.$$

Then use the representation $A = Q^T DQ$, where $D$ is diagonal and $Q$ is orthogonal.

56. Use Example 55 to complete the proof of the following theorem:

If a Runge-Kutta method is algebraically stable, then it is $B$-stable.

57. Consider a linear multistep method

$$\alpha_k\, u_{j+k} + \alpha_{k-1}\, u_{j+k-1} + \ldots + \alpha_0 u_j = \tau\,(\beta_k\, f_{j+k} + \beta_{k-1}\, f_{j+k-1} + \ldots + \beta_0 f_j)$$

with generating polynomials $\rho(z)$ and $\sigma(z)$, which have no common root. Show that a point $\mu$ on the boundary $\partial S$ of the stability domain $S$ can be represented in the following way:

$$\mu = \frac{\rho(\zeta)}{\sigma(\zeta)} \quad \text{with } \zeta = e^{i\theta}$$

for some $\theta \in [0, 2\pi)$.

58. Show that the 2-step BFD method

$$\frac{3}{2}u_{j+2} - 2u_{j+1} + \frac{1}{2}u_j = \tau\, f(t_{j+2}, u_{j+2})$$

is $A$-stable.

Hint: Use Example 57 to show $\operatorname{Re}\mu \geq 0$ for all points $\mu \in \partial S$.

59. Show that the 2-step BFD method

$$\frac{3}{2}u_{j+2} - 2u_{j+1} + \frac{1}{2}u_j = \tau\, f(t_{j+2}, u_{j+2})$$

is $G$-stable.

Hint: For

$$\frac{3}{2}\hat{u}_{j+2} - 2\hat{u}_{j+1} + \frac{1}{2}\hat{u}_j = \tau\, f(t_{j+2}, \hat{u}_{j+2})$$

and $\Delta u_l = \hat{u}_l - u_l$, show first

$$E = \left(\frac{3}{2}\Delta u_{j+2} - 2\Delta u_{j+1} + \frac{1}{2}\Delta u_j, \Delta u_{j+2}\right) \leq 0.$$

Next, find a $2 \times 2$ matrix $G$ and coefficients $a_0$, $a_1$, $a_2$ such that

$$E = \|\Delta U_{j+1}\|_G^2 - \|\Delta U_j\|_G^2 + (a_2 \Delta u_{j+2} + a_1 \Delta u_{j+1} + a_0 \Delta u_j)^2,$$

by comparing the coefficients of $(\Delta u_l, \Delta u_m)$ for $l, m \in \{j, j+1, j+2\}$.