ÜBUNGEN ZU NUMERIK ZEITABHÄNGIGER PROBLEME

für den 7.1.2008

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48. Let $R(z) = P(z)/Q(z)$ be the stability function of an s-stage Runge-Kutta method. Show: The Runge-Kutta method is A-stable if and only if

    (a) the Runge-Kutta method is I-stable, i.e.: $|R(iy)| \leq 1$ for all $y \in \mathbb{R}$, and

    (b) $R(z)$ is complex differentiable in $\mathbb{C}^-$.

    Hint: Use (without proof) the maximum principle in $\mathbb{C}$: Let f be a complex differentiable function on a connected and open set $D \subset \mathbb{C}$. Then $|f|$ cannot attain its maximum in D unless f is a constant function.

    Apply this principle to R and the domain $D = \{z \in \mathbb{C} : \operatorname{Re} z < 0 \text{ and } |z| < r\}$ for sufficiently large r, conclude that the maximum of $|R|$ is attained on ∂D. $R(\infty) = \lim_{|z|\to\infty} R(z)$ exists and $|R(\infty)| \leq 1$.

49. Let $R(z) = P(z)/Q(z)$ be the stability function of an s-stage Runge-Kutta method. Show: The Runge-Kutta method is I-stable if and only if

    (c) $E(y) \geq 0$ for all $y \in \mathbb{R}$ with $E(y) = |Q(iy)|^2 - |P(iy)|^2$.

    Show that $E(y)$ is an even polynomial of degree $\leq 2\max(k, j)$, where k and j denote the degree of $P(z)$ and $Q(z)$, respectively.

    Hint: Show and use $|Q(iy)|^2 = Q(iy)Q(-iy)$ and $|P(iy)|^2 = P(iy)P(-iy)$.

50. Assume that $R(z) = P(z)/Q(z)$ is the stability function of a Runge-Kutta method of order p. Show
$$E(y) = O(y^{p+1}) \quad \text{for } y \to 0.$$
    Hint: Show and use
$$|e^z| - \frac{|P(z)|}{|Q(z)|} = O(z^{p+1}).$$
    Set $z = iy$ and conclude $|Q(iy)| - |P(iy)| = O(y^{p+1})$.

51. Assume that $R(z) = P(z)/Q(z)$, where $P(z)$ is a polynomial of degree k and $Q(z)$ is a polynomial of degree j, is the stability function of a Runge-Kutta method of order $p \geq 2j-2$. Show that the Runge-Kutta method is I-stable if and only if $|R(\infty)| \leq 1$.

    Hint: Show first that $E(y)$ is of the form $K \cdot y^{2j}$ by using Examples 49 and 50. Then show: $K \geq 0$ is equivalent to $|R(\infty)| \leq 1$.

52. Let $c_1 = 0$, c_2, $\ldots$, c_{s-1}, $c_s = 1$ and b_1, $\ldots$, b_s be the coefficients of the correspond Lobatto quadrature rule. It can be shown that the c_i are all distinct and the b_i are all non-zero. The coefficients a_{ij} of the Lobatto IIIA method are uniquely determined by the conditions $C(s)$. Show

$$a_{1j} = 0 \quad \text{and} \quad a_{sj} = b_j \quad \text{for all } j = 1, \ldots, s.$$

It can be shown that the Lobatto IIIA method is of order $p = 2s - 2$. Show that the stability function of the Lobatto IIIA method is the $(s-1, s-1)$-Padé approximation.

Hint for the second part: $R(z) = \det(I - z(A - eb^T))/\det(I - zA)$. The first row of A and the last row of $A - eb^T$ vanish.

53. Consider the implicit midpoint rule:

$$\begin{aligned} g_1 &= u_\tau(t) + \frac{1}{2}\tau f(t + \frac{1}{2}\tau, g_1), \\ u_\tau(t+\tau) &= u_\tau(t) + \tau f(t + \frac{1}{2}\tau, g_1). \end{aligned}$$

$u_{\tau/2}(t+\tau)$ denotes the result of two steps of the implicit midpoint rule with step size $\tau/2$ starting at $(t, u_\tau(t))$. Consider the extrapolation method

$$\hat{u}_\tau(t+\tau) = u_{\tau/2}(t+\tau) + \frac{1}{3}(u_{\tau/2}(t+\tau) - u_\tau(t+\tau)).$$

Determine the stability functions of the implicit midpoint rule and the corresponding local extrapolation method. Compute the value of the stability functions at ∞.