ÜBUNGEN ZU NUMERIK ZEITABHÄNGIGER PROBLEME

für den 14.12.2007

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43. Consider the polynomial

$$\rho(z) = z^k + \alpha_{k-1}\, z^{k-1} + \cdots + \alpha_0$$

and the matrix

$$A = \begin{pmatrix} -\alpha_{k-1} & -\alpha_{k-2} & \dots & -\alpha_0 \\ 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}.$$

Show that the roots $\zeta \in \mathbb{C}$ of $\rho(z)$ are the eigenvalues of A and vice versa. Determine all eigenvectors and the geometric multiplicity of the eigenvalues. Show that the following two statements are equivalent:

(a) There exists a constant C such that

$$\|A^j\| \le C \quad \text{for all } j \in \mathbb{N}.$$

(b) Each eigenvalue λ of A satisfies one of the two conditions:

i. $|\lambda| < 1$

ii. $|\lambda| = 1$ and λ is a simple eigenvalue.

(An eigenvalue λ is called simple if the algebraic multiplicity of λ is 1.)

44. Determine the stability function $R(z)$ and the stability domain S of the θ-method

$$u_1 = u_0 + \tau\big[(1-\theta) f(t_0, u_0) + \theta f(t_1, u_1)\big]$$

for $\theta \in [0, 1]$. Describe the stability domain graphically.

45. Find a polynomial of the form

$$R(z) = 1 + z + a\, z^2$$

such that the stability domain on the negative real axis is as large as possible, i.e.:

$$|R(x)| \le 1 \quad \text{for } -\alpha \le x \le 0$$

with $\alpha > 0$ as large as possible.

46. Find a polynomial of the form

$$R(z) = 1 + z + b\,z^2$$

such that the stability domain on the imaginary axis is as large as possible, i.e.:

$$|R(iy)| \leq 1 \quad \text{for } -\beta \leq y \leq \beta$$

with $\beta > 0$ as large as possible.

47. Let $R(z)$ be the stability function of a Runge-Kutta method and $R^*(z)$ the stability function of its adjoint method. Show

$$R^*(z) = \frac{1}{R(-z)}.$$

Hint: Apply the Runge-Kutta method to the model problem $u' = \lambda\,u$. Then replace τ by $-\tau$ and t by $t + \tau$ in order to obtain the adjoint method.