ÜBUNGEN ZU NUMERIK ZEITABHÄNGIGER PROBLEME

für den 10.12.2007

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37. Find the multistep method of the form

$$u_{j+2} + \alpha_1\, u_{j+1} + \alpha_0\, u_j = \tau\,(\beta_1\, f_{j+1} + \beta_0\, f_j)$$

of the highest possible order.

38. Consider the 2-step method

$$u_{j+2} + 4\, u_{j+1} - 5\, u_j = \tau\,(4\, f_{j+1} + 4\, f_j),$$

applied to the initial value problem

$$u'(t) = u(t), \quad u(0) = 1.$$

Find an analytic expression for u_j for $u_0 = 1$ and arbitrary u_1.

Hint: Here we have $f_{j+1} = u_{j+1}$ and $f_j = u_j$. Find the general solution of the resulting linear recurrence relation.

39. Show for all matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{q\times r}$, $C \in \mathbb{R}^{n\times p}$, and $D \in \mathbb{R}^{r\times s}$:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

40. Let $A \in \mathbb{R}^{n\times n}$. Show that the following two statements are equivalent:

   (a) There exists a constant C such that

   $$\|A^j\| \leq C \quad \text{for all } j \in \mathbb{N}.$$

   (b) Each eigenvalue λ of A satisfies one of the two conditions:

      i. $|\lambda| < 1$

      ii. $|\lambda| = 1$ and the geometric multiplicity of λ is equal to its algebraic multiplicity.

41. Let

$$\begin{aligned} \Phi_i^*(j) &= \prod_{l=0}^{i-1}(t_{j+1} - t_{j-l})\; f[t_j, t_{j-1}, \ldots, t_{j-i}] \\ \Phi_i(j) &= \prod_{l=0}^{i-1}(t_j - t_{j-l-1})\; f[t_j, t_{j-1}, \ldots, t_{j-i}] \end{aligned}$$

and

$$\beta_i(j) = \prod_{l=0}^{i-1} \frac{t_{j+1} - t_{j-l}}{t_j - t_{j-l-1}}$$

Show the following recurrence relations:

$$\begin{gathered} \beta_0(j) = 1, \\ \Phi_0(j) = \Phi_0^*(j) = f_j, \\ \beta_i(j) = \beta_{i-1}(j)\ \frac{t_{j+1} - t_{j-i+1}}{t_j - t_{j-i}}, \\ \Phi_i(j) = \Phi_{i-1}(j) - \Phi_{i-1}^*(j-1), \quad \Phi_i^*(j) = \beta_i(j)\Phi_i(j). \end{gathered}$$

42. Show that the variable step size BDF method is of the form

$$\sum_{i=1}^{k} \tau_j \prod_{l=1}^{i-1}(t_{j+1} - t_{j-l+1})\ u[t_{j+1}, t_j, \ldots, t_{j-i}] = \tau_j f_{j+1},$$

where the divided differences $u[t_{j+1}, t_j, \ldots, t_{j-i}]$ are recursively defined by

$$\begin{aligned} u[t_j] &= u_j, \\ u[t_{j+1}, t_j, \ldots, t_{j-l}] &= \frac{u[t_{j+1}, t_j, \ldots, t_{j-l+1}] - u[t_j, t_{j-1}, \ldots, t_{j-l}]}{t_{j+1} - t_{j-l}}. \end{aligned}$$