ÜBUNGEN ZU NUMERIK ZEITABHÄNGIGER PROBLEME

für den 12.11.2007

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20. Consider a one-step method
$$u_{j+1} = u_j + \tau_j\, \phi(t_j, u_j, \tau_j) \quad \text{for } j = 0, \dots, m-1.$$
for solving the initial value problem:
$$\begin{aligned} u'(t) &= f(t, u(t)), \quad t \in I = [t_0, T], \\ u(t_0) &= u_0. \end{aligned}$$
Assume that
$$\|\phi(t, w, \tau) - \phi(t, v, \tau)\| \le \Lambda\, \|w - v\| \quad \text{for all } t, v, w \text{ and all } \tau.$$
Show the following estimate of the global error in terms of the consistency error:
$$\max_{j=0,\dots,m-1} \frac{1}{\tau_j} \|e_{j+1} - e_j\| \le e^{\Lambda(T-t_0)}\, \|\psi_\tau(u)\|_{X_h}.$$
Hint: Show and use
$$\frac{1}{\tau_j} \|e_{j+1} - e_j\| \le \Lambda\, \|e_j\| + \|\psi_{j+1}(u)\|.$$

21. Consider a Runge-Kutta method with coefficients A, b, c for solving the initial value problem:
$$\begin{aligned} u'(t) &= f(t, u(t)), \quad t \in I = [t_0, T], \\ u(t_0) &= u_0. \end{aligned}$$
Assume that
$$\|f(t, w) - f(t, v)\| \le L\, \|w - v\| \quad \text{for all } t, v, w.$$
Show:
$$\|\phi(t, w, \tau) - \phi(t, v, \tau)\| \le \Lambda\, \|v - w\| \quad \text{for all } t, v, w$$
and sufficiently small $\tau > 0$ with
$$\Lambda = L |b|^T (I - \tau L\, |A|)^{-1} e,$$
where $|b| = (|b_i|)_{i=1,\dots,s}$, $|A| = (|a_{ij}|)_{i,j=1,\dots,s}$, $e = (1, \dots, 1)^T \in \mathbb{R}^s$ and $I \in \mathbb{R}^{s \times s}$ is the identity matrix.
Hint: Show and use
$$\|\phi(t, w, \tau) - \phi(t, v, \tau)\| \le L \sum_{i=1}^{s} |b_i| \|g_i^{(2)} - g_i^{(1)}\|,$$
$$\|g_i^{(2)} - g_i^{(1)}\| \le \|w - v\| + \tau L \sum_{j=1}^{s} |a_{ij}| \|g_j^{(2)} - g_j^{(1)}\|.$$

22. Assume the notations and assumptions of Exercise 21. Show for the classical 4-stage Runge-Kutta method:

$$1 + \tau\Lambda \leq e^{\tau L}.$$

Hint: Show $1 + \tau\Lambda = R(\tau L)$ with $R(z) = 1 + zb^T(I - zA)^{-1}e$. Determine $R(z)$ and show $R(z) \leq e^z$ for $z \in \mathbb{R}$ with $z \geq 0$.

23. Let f be a continuous function that satisfies the Lipschitz condition

$$\|f(t, w) - f(t, v)\| \leq L\,\|w - v\| \quad \text{for all } t, v, w.$$

Let $v(t)$ and $w(t)$ be solutions of the differential equation

$$u'(t) = f(t, u(t)) \quad t \in [t_0, T]$$

with initial values $v(t_0) = v_0$ and $w(t_0) = w_0$. Show:

$$\|w(t) - v(t)\| \leq e^{L(t-t_0)}\,\|w_0 - v_0\| \quad \text{for } t \in [t_0, T].$$

Hint: Use the corresponding estimates for the Euler polygons provided by Lemma 5.1.

24. Let f be a continuous function that satisfies the Lipschitz condition

$$\|f(t, w) - f(t, v)\| \leq L\,\|w - v\| \quad \text{for all } t, v, w.$$

Let $v(t)$ and $w(t)$ be solutions of the differential equation

$$u'(t) = f(t, u(t)) \quad t \in [t_0, T]$$

with initial values $v(t_0) = v_0$ and $w(t_0) = w_0$. Show:

$$\|w(t) - v(t) - (w_0 - v_0)\| \leq \left(e^{L(t-t_0)} - 1\right)\,\|w_0 - v_0\| \quad \text{for } t \in [t_0, T].$$

Hint: Show for the corresponding Euler polygons:

$$\|(w_{j+1} - v_{j+1}) - (w_j - v_j)\| \leq \tau_j\,L\,\|w_j - v_j\| \leq \tau_j\,L\,e^{L(t_j-t_0)}\,\|w_0 - v_0\|.$$