T U T O R I A L

"Numerical Methods for Solving Partial Differential Equations"

for the Lectures on NuPDE

T XI Monday, 28 January 2008 (Time: 08:30 – 10:00, Room: T 212)

2.3 Runge-Kutta methods for IVPs of ODEs

61 If we evaluate the integral $\int_t^{t+\tau} f(s,\, u(s))\, ds$ using the trapezoidal rule (TR), i. e.,

$$\int_t^{t+\tau} f(s,\, u(s))\, ds \overset{\text{TR}}{\approx} \frac{\tau}{2}\big[f(u,\, u(t)) + f(t+\tau,\, u(t+\tau))\big]\,,$$

and approximate $u(t+\tau) \approx u(t) + \tau\, f(t,\, u(t))$ according to the forward Euler method, we obtain the so-called *Heun method*. This method corresponds to the Butcher table

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

from which you can see that the Heun method is a 2-stage Runge-Kutta method. The scheme reads

$$u_{j+1} = u_j + \frac{\tau}{2}\big[f(t_j, u_j) + f(t_{j+1}, u_j + \tau f(t_j,\, u_j))\big]\,.$$

Considering a Taylor expansion of the local error $u(t+\tau) - u_\tau(t+\tau)$, show that the Heun method has consistency order 2.

For programming the following exercises you are allowed to use `matlab`*, C, C++ or anything you like. Please, save or print out your plots.*

62 Consider the initial-value problem

$$\begin{aligned} u'(t) &= B\,(u + A) \qquad 0 < t < 1\,, \\ u(0) &= 0\,, \end{aligned} \tag{2.33}$$

with $B = 3$ and $A = e^B - 1$.

(a) Compute analytically the exact solution u of (2.33).

(b) Consider the discretization step sizes $\tau = 1,\ 1/2,\ 1/4,\ 1/8,\ 1/16$, and compute the numerical approximation u_τ of u using the following explicit Runge-Kutta methods:

- the (explicit) forward Euler method (FE)
- the Heun method (H)

– the classical fourth-order Runge-Kutta method (RK4):

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array} \tag{2.34}$$

Plot the exact and the numerical solution for each τ_i.

(c) Fill in the following table reporting the approximation $u_\tau(1)$ of $u(1)$:

method		approx. for $\tau =$ 1	1/2	1/4	1/8	1/16	u(1)
FE	(*)						
	(E)					—	
H	(*)						
	(E)					—	
RK4	(*)						
	(E)					—	

Here,

(*) denotes the 'original' method

(E) corresponds to a post-processing of the computed values using a global extrapolation strategy considering the formula

$$\widehat{u}_\tau(t) = u_{\tau/2}(t) + \frac{u_{\tau/2}(t) - u_\tau(t)}{2p - 1},$$

p being the order of the considered method.

$\boxed{63}$ Consider the following initial-value problem

$$\begin{aligned} u'(t) &= -50\big(u(t) - \cos(t)\big) \qquad 0 < t < 1.5\,, \\ u(0) &= 0\,. \end{aligned} \tag{2.35}$$

(a) Compute analytically the exact solution u of (2.35).

(b) Consider the discretization steps sizes $\tau = \frac{1}{20}, \frac{3}{80}, \frac{1}{30}, \frac{1}{40}$, and compute the numerical approximation of u_τ of u using the following explicit Runge-Kutta methods:

– the (explicit) forward Euler method (FE)
– the Heun method (H)
– the (implicit) backward Euler method (BE), i. e., the scheme

$$u_{j+1} = u_j + \tau f(t_{j+1}, u_{j+1})\,.$$

Plot the exact and the computed numerical solutions and compare them. Moreover, fill in the following table with the approximations $u_\tau(1.5)$ of the exact solution u at $t = 1.5$:

method	approx. for $\tau =$ 1/20	3/80	1/30	1/40	u(1.5)
FE					
H					
BE					

c) Start the explicit forward Euler method from $t_0 = 1/2$ using the exact solution $u(t_0) = u(1/2)$ and the discretization step size $\tau = 1/20$. Compare graphically the exact and the numerical solution obtained in that case. Which value of τ must be considered in order to guarantee that the method is stable?

d) Compute the solution of (2.35) considering the (implicit) backward Euler method and the discretization step size $\tau = 1/2$. Compare the computed solution and the exact one graphically.

$\boxed{64}$ The orbit of a satellite in the plane of the Earth-Moon system can be modeled by the following second-order ODE system:

$$\begin{aligned} y_1'' &= y_1 + 2\,y_2' - (1-\mu)\,\frac{y_1+\mu}{D_1} - \mu\,\frac{y_1-(1-\mu)}{D_2}\,, \qquad 0<t<T\,,\\ y_2'' &= y_2 - 2\,y_1' - (1-\mu)\,\frac{y_2}{D_1} - \mu\,\frac{y_2}{D_2}\,, \qquad 0<t<T\,,\\ y_1(0) &= 0.994\,, \qquad y_1'(0) = 0\,,\\ y_2(0) &= 0\,, \qquad y_2'(0) = -2.001\,585\,106\,379\,082\,522\,405\,378\,622\,24\,, \end{aligned} \tag{2.36}$$

where

$$D_1 = \left[(y_1+\mu)^2 + y_2^2\right]^{3/2}, \quad D_2 = \left[(y_1-(1-\mu))^2 + y_2^2\right]^{3/2}, \quad \mu = 0.012277471\,.$$

For the given data, the system has a solution whose period is

$$t_{\text{per}} = t = \underline{17.065\,216\,560\,157\,96}2\,558\,891\,720\,624\,9\,.$$

a) Rewrite (2.36) in an equivalent form as a system of first-order ODEs.

b) Solve (2.36) numerically considering
   - the classical fourth-order Runge-Kutta method (2.34)
   - another method of your choice.

c) Plot the computed trajectory $(y_{1\tau}(t),\, y_{2\tau}(t))$, considering a linear interpolation of the computed positions.