T U T O R I A L

"Numerical Methods for Solving Partial Differential Equations"

for the Lectures on NuPDE

T IX Monday, 14 January 2008 (Time: 08:30 – 10:00, Room: T 212)

1.11 Optimal Preconditioning (MDS)

Finish exercises 49 and 50 from tutorial VIII.

51 Consider the elliptic problem: Find $u(x)$ such that

$$\begin{aligned} -u''(x) &= 8 \qquad x \in (0,\,1)\,, \\ u(x) &= -1 \qquad x = 0\,, \\ u(x) + u'(x) &= -5 \qquad x = 1\,, \end{aligned} \tag{1.27}$$

and its corresponding variational formulation

$$\text{find } u \in V_0: \qquad a(u,\,v) = \langle F,\,v \rangle \qquad \forall v \in V_0\,. \tag{1.28}$$

Solve the system

$$K_h\,\underline{u}_h = \underline{f}_h\,,$$

which originates from the FEM discretization of (1.28), by the preconditioned conjugate gradient method with the following strategies:

a) without any preconditioner ($C = I$),
b) with the Jacobi preconditioner ($C = \mathrm{diag}\,(K_h)$),
c) with the MDS preconditioner on two levels ($L = 2$),
d) with the MDS preconditioner on four levels ($L = 4$).

Let h_f denote the mesh size of the **FINEST** grid in the whole computation, all the coarser grids are nested. Fill in the following table:

Number of iterations

	$C = I$	Jacobi	MDS ($L = 2$)	MDS ($L = 4$)
$h_f = 1/800$				
$h_f = 1/1600$				

optional: *CPU time*

	$C = I$	Jacobi	MDS ($L = 2$)	MDS ($L = 4$)
$h_f = 1/800$				
$h_f = 1/1600$				

How do the number of iterations (and the CPU time) depend on the mesh size and the number of unknowns?

2 Parabolic Differential Equations

2.1 Solution methods for IVPs of ODEs

We consider the model problem

$$\begin{aligned} \frac{\partial}{\partial t}u(x,\,t) - \frac{\partial^2}{\partial x^2}u(x,\,t) &= f(x,\,t) \qquad t \in (0,\,T), \quad x \in (0,\,1)\,, \\ u(x,\,0) &= u_0(x) \qquad x \in (0,\,1)\,, \end{aligned} \tag{2.29}$$

for some $T > 0$. Furthermore we consider the semi-discretization $(\underline{6})_h$ from section 2.2.2 in the lecture notes, i. e.,

$$\begin{aligned} M_h\,\underline{u}_h'(t) + K_h\,\underline{u}_h(t) &= \underline{f}_h(t) \qquad t \in (0,\,T)\,, \\ M_h\,\underline{u}_h(0) &= \underline{g}_h\,. \end{aligned} \tag{2.30}$$

$\boxed{52}$ Compute the condition number $\kappa(M_h^{-1}K_h)$.

Hint: $M_h^{-1}K_h$ is not symmetric but self-adjoint (in which scalar product?)

$\boxed{53}$ Transform the first equation of (2.30) into the form

$$\underline{u}_h'(t) = G(t,\,\underline{u}_h)\,,$$

and show that $G : [0,\,T) \times \mathbb{R}^N$ is Lipschitz continuous in the second component, i. e., there exists $L > 0$ with

$$\|G(t,\,x) - G(t,\,y)\|_{\ell_2} \leq L\|x - y\|_{\ell_2}\,.$$

Determine the Lipschitz constant L.

$\boxed{54}$ Show that system (2.30) has a unique solution $\underline{u}_h \in [\mathcal{C}(0,\,1)]^N$.

Hint: Use the following theorem.

THEOREM 2.1 (Picard-Lindelöf) *Let X denote the Banach space $\mathbb{R}^N$ with the Euclidean norm. Moreover, let $G : [0,\,T] \times X \to X$ a continuous mapping which is Lipschitz continuous in the second component. Then, for each $u_0 \in X$ there exists a unique and continuous solution $u : [0,\,T] \to X$ to the abstract initial value problem*

$$\begin{aligned} \frac{d}{dt}u(t) &= G(t,\,u(t)) \qquad t \in (0,\,T)\,, \\ u(0) &= u_0\,. \end{aligned}$$