T U T O R I A L

"Numerical Methods for Solving Partial Differential Equations"

to the Lectures on NuPDE

T VI Monday, 3 December 2007 (Time: 08:30 – 10:00, Room: T 212)

1.6 Error estimates

32 Consider the problem

$$\text{find } u \in V_0 \cap H^1(0,1): \quad a(u,v) = \langle F, v\rangle \qquad \forall v \in V_0\,, \tag{1.22}$$

with $V_0 = \{v \in H^1(0,1) : v(0) = 0\}$, where the bilinear form is coercive and bounded, i. e.,

$$a(v,v) \ge \mu_1 \|v\|_1^2\,, \qquad a(u,v) \le \mu_2 \|u\|_1 \|v\|_1 \qquad \forall u, v \in V_0\,.$$

Furthermore, let $\{u_h\}$ be a family of conforming finite element approximation of u based on Courant elements on a uniform mesh with mesh size h. Show that we have at least convergence in $H^1(0,1)$ for $h \to 0$:

$$\lim_{h\to 0} \|u - u_h\|_1 \le \frac{\mu_2}{\mu_1} \lim_{h\to 0} \inf_{v_h \in V_h} \|u - v_h\|_1 = 0$$

Hint: Use that $H^2(0,1)$ is dense in $H^1(0,1)$.

33 Consider the model problem (1.22) as before with $a(u,v) = \int_0^1 u'(x)v'(x)\,dx$ and $\langle F, v\rangle = \int_0^1 f(x)v(x)\,dx$, and consider a finite element approximation u_h based on linear Courant elements. Let μ_1 be the coercivity constant of $a(\cdot,\cdot)$. Then, the following a-posteriori error estimate holds (see also your lecture notes):

$$\|u - u_h\|_1 \le \frac{C}{\mu_1}\, \eta(u_h)\,, \tag{1.23}$$

for a positive constant $C > 0$, where

$$\eta(u_h) = \left(\sum_{k=1}^{N_h} \eta_k^2\right)^{1/2}, \qquad \eta_k = h_k \|f\|_{L_2(T_k)}\,.$$

Write a function `ErrorEstimator(↓mesh, ↓(*f)(x), ↑error)` that computes the error estimator `error`$=\eta$ for a given function `f`$=f$.

34 Write a function `ImplementNeumannBC(↓i, ↓g, ↕vector)` to implement the Neumann boundary condition

$$\pm u'(x_i) = g_N(x_i)$$

for `i`=i, `g`=$g_N(x_i)$, and the load vector `vector` (note, that the sign depends on whether i is the right or left boundary point). Use this to compute the solution of the boundary value problem

$$\begin{aligned} u''(x) &= f(x) \qquad x \in (0,\,1) \\ u(0) &= -1 \\ u'(1) &= -3 \end{aligned} \tag{1.24}$$

for $f(x) = 8$ with the Richardson method that you implemented in Tutorial V, and estimate the error with your function `ErrorEstimator`.

1.7 Schwarz methods

In the following we consider the boundary value problem

$$u''(x) = f(x) \qquad x \in (0,\,1)\,,$$

with $u(0) = u(1) = 0$ and discretize by FEM with N Courant elements, not necessarily on a uniform mesh. We denote the $(N-1) \times (N-1)$ FEM stiffness matrix by K.

$\boxed{35}$ Consider the additive Schwarz method (ASM) with the space splitting

$$V = \sum_{s=1}^{N-1} V_s\,, \qquad V_s := \operatorname{span}\{\varphi_s\}\,,$$

where φ_k is the Courant basis function at the node x_k. Have a look to Algorithm 1.29 in your lecture notes, figure out what R_s is, and compute

$$C_s := R_s\,K\,R_s^\top$$

for $s = 1, \ldots, N-1$ explicitly.

$\boxed{36}$ For the same setting compute

$$R_s^\top C_s^{-1}\,R_s$$

(for $s = 1, \ldots, N-1$) and

$$C^{-1} := \sum_{s=1}^{N-1} R_s^\top C_s^{-1}\,R_s$$

explicitly.

1.8 Dirichlet boundary condition via penalty technique

$\boxed{37}$ Consider the boundary value problem (1.24) but change the Dirichlet boundary condition to the following Robin-type boundary condition

$$-u'(0) = \alpha\big(g_D(0) - u(0)\big)\,,$$

with a large α. Like before $g_D(0) = -1$ and $f(x) = 8$.

Use your program to compute the FEM solution $u_h^{(\mathrm{R})}$ with the Robin boundary condition, and the solution $u_h^{(\mathrm{D})}$ with the original Dirichlet boundary condition using `ImplementDirichletBC`. In the Richardson iteration chose ε=`tol`=10^{-10}. Try 30 elements and the cases $\alpha = 10^3$, 10^4, 10^5 and 10^6. Compare the solutions $u_h^{(\mathrm{R})}$ and $u_h^{(\mathrm{D})}$, e. g., compute $\|\underline{u}_h^{(\mathrm{D})} - \underline{u}_h^{(\mathrm{R})}\|_{\ell_2}$.