# T U T O R I A L

## "Numerical Methods for Solving Partial Differential Equations"

to the Lectures on NuPDE

**T II** Monday, 29 October 2007 (Time: 08:30 – 10:00, Room: T 212)

### 1.2 BVPs for second-order PDEs

**07** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with the boundary $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$. Find the variational formulations ($V = ?, V_0 = ?, V_g = ?, a(\cdot,\cdot) = ?, \langle F, \cdot\rangle = ?$) for the following boundary value problem: Find $u : \Omega \to \mathbb{R}$ with

$$-\mathrm{div}(a(x)\nabla u(x)) + b(x)\cdot\nabla u(x) + c(x)u(x) = f(x)\,, \qquad x \in \Omega\,, \tag{1.11}$$

$$u(x) = g_D(x)\,, \qquad x \in \Gamma_D, \tag{1.12}$$

$$a(x)\frac{\partial u}{\partial n}(x) = g_N(x)\,, \qquad x \in \Gamma_N\,, \tag{1.13}$$

$$a(x)\frac{\partial u}{\partial n}(x) = \alpha(x)(g_R(x) - u(x))\,, \qquad x \in \Gamma_R\,. \tag{1.14}$$

where $f$, $g_D$, $g_N$, $g_R$ and $c$ map to $\mathbb{R}$, and $b : \Omega \to \mathbb{R}^d$.

**08** Show that the variational problem of the BVP (1.11)–(1.14) has a unique solution provided that the following conditions imposed on the data are fulfilled:

(1) $a \in L_\infty(\Omega)$ with $0 < a_1 = \text{const} \le a(x) \le a_2 = \text{const}$ for almost all $x \in \Omega$,

(2) $c \in L_\infty(\Omega)$ with $0 < c_1 = \text{const} \le c(x) \le c_2 = \text{const}$ for almost all $x \in \Omega$,

(3) $b \equiv 0$,

(4) $f \in L_2(\Omega)$,

(5) There exists a function $g \in H^1(\Omega)$ with $\gamma_D g := g_{|\Gamma_D} = g_D$.

(6) $g_N \in L_2(\Gamma_N)$, $g_R \in L_2(\Gamma_R)$,

(7) $\alpha \in L_\infty(\Gamma_R)$ and $0 \le \alpha(x) \le \alpha_2 = \text{const}$ for almost all $x \in \Gamma_R$.

*Hint: You will need the trace inequality (lecture notes, section "trace operator")*

**08*** Show **08** but replacing assumption (3) by

(3*) $b \in L_\infty(\Omega)^d$ with $|b(x)| \le b_2 = \text{const}$ and there exists $\varepsilon > 0$ such that

$$a_1 - \frac{1}{2\varepsilon}b_2 > 0 \qquad \text{and} \qquad c_1 - \frac{\varepsilon}{2}b_2 > 0\,.$$

*Hint: For the proof of the $V_0$-ellipticity use the $\varepsilon$-inequality*

$$a\,b \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \qquad \forall a,\, b \in \mathbb{R} \quad \forall \varepsilon > 0$$

*and the elementary inequality* $a + b \le \sqrt{2(a^2+b^2)} \qquad \forall a,\, b \in \mathbb{R}$.

$\boxed{09}$ Due to Corollary 1.8 the solution of the variational problem of the BVP (1.11)–(1.14) can be approximated by the fixed point iteration (18)=(19) given in the lectures. Give the classical formulation of this fixed point iteration for the variational problem derived in Exercise $\boxed{07}$ (including the boundary conditions).

$\boxed{10}$ Show that the variational problem: find $u \in V = V_g = V_0 = H^1(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \Big( \int_\Omega u(x)\, dx \Big) \Big( \int_\Omega v(x)\, dx \Big) = \int_\Omega f(x) v(x)\, dx \quad \forall v \in V\,, \tag{1.15}$$

has a unique solution for given $f \in L_2(\Omega)$. Furthermore, if the right-hand side fulfills the solvability condition

$$\langle F,\, c\rangle := \int_\Omega f(x)\, c\, dx = 0 \quad \forall c \in \mathbb{R}, \tag{1.16}$$

for the Neumann problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega\,, \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \Gamma\,, \end{aligned} \tag{1.17}$$

then the solution $u$ of (1.15) is also a weak solution of the Neumann problem (1.17), and satisfies the orthogonality condition

$$\int_\Omega u(x)\, dx = 0\,. \tag{1.18}$$

$\boxed{11}$ Let $\Omega = (0,1) \times (0,1)$ and $\Gamma_D = [0,1] \times \{0\}$. Show the Friedrichs inequality in a constructive way, i.e. determine an explicit constant $c_P > 0$ such that

$$\|v\|_0 \le c_P\, |v|_1 \quad \forall v \in V_0 = \{ v \in H^1(\Omega) : v = 0 \quad \text{on } \Gamma_D \}\,. \tag{1.19}$$

$\boxed{12}$ Let $\widetilde{\Gamma} \subset \Gamma = \partial\Omega$ with $\operatorname{meas}(\widetilde{\Gamma}) = \int_{\widetilde{\Gamma}} ds > 0$. Show the equivalence of the norm

$$\|v\|_1^* := \Big( \int_{\widetilde{\Gamma}} (v(x))^2 ds + |v|_1^2 \Big)^{1/2} \tag{1.20}$$

with the standard norm $\|v\|_1$ in $H^1(\Omega)$.
*Hint: Use Sobolev's norm equivalence theorem.*