T U T O R I A L

"Numerical Methods for Solving Partial Differential Equations"

to the Lectures on NuPDE

T I Monday, 22 October 2007 (Time: 08:30 – 10:00, Room: T 212)

1 Elliptic Differential Equations

1.1 BVPs for Second-order ODEs

01 Find the variational formulations ($V = ?, V_0 = ?, V_g = ?, a(\cdot, \cdot) = ?, \langle F, \cdot \rangle = ?$) for the following boundary value problems:

(a)
$$\begin{aligned} &-u''(x) + u(x) = f(x), \quad \text{for } x \in (0,1),\\ &u(0) = g_0,\\ &u(1) = g_1. \end{aligned}$$

(b)
$$\begin{aligned} &-u''(x) + u(x) = f(x), \quad \text{for } x \in (0,1),\\ &u'(0) = g_0,\\ &-u'(1) = g_1. \end{aligned}$$

(c)
$$\begin{aligned} &-u''(x) + u(x) = f(x), \quad \text{for } x \in (0,1),\\ &u'(0) = \alpha_0 u(0) - \beta_0 g_0,\\ &-u'(1) = \alpha_1 u(1) - \beta_1 g_1. \end{aligned}$$

02 Find the variational formulations ($V = ?, V_0 = ?, V_g = ?, a(\cdot, \cdot) = ?, \langle F, \cdot \rangle = ?$) for the following boundary value problem:

$$-\bar{a}(x)u''(x) + \bar{b}(x)u'(x) + \bar{c}(x)u(x) = f(x) \qquad \text{for } x \in (0,1), \tag{1.1}$$
$$u'(0) = \alpha_0(u(0) - g_0), \tag{1.2}$$
$$-u'(1) = \alpha_1(u(1) - g_1), \tag{1.3}$$

where $\bar{a}(\cdot), \bar{b}(\cdot), \bar{c}(\cdot), f(\cdot)$ are given, sufficiently smooth functions, and $\alpha_0, \alpha_1, g_0, g_1 \in \mathbb{R}$ are also given.
Hint: Rewrite first the differential operator of the differential equation (1.1) in the so-called divergence form $-(a(x)u'(x))' + b(x)u'(x) + c(x)u(x)$!

03 Derive the variational formulation

$$\text{find } u \in V_g: \quad a(u, v) = \langle F, v \rangle \quad \forall v \in V_0 \tag{1.4}$$

of the Neumann boundary value problem

$$-u''(x) = f(x) \quad \text{for } x \in (0,1), \tag{1.5}$$
$$u'(0) = g_0, \tag{1.6}$$
$$-u'(1) = g_1, \tag{1.7}$$

and show that the following statements are valid:

(a) If u is a solution of (1.4), then, for any constant $c \in \mathbb{R}$, $u + c$ is also a solution !

(b) If (1.4) has a solution, then

$$\langle F, c \rangle = 0, \quad \forall c \in \mathbb{R}. \tag{1.8}$$

$\boxed{04}$ Show that the variational problem: find $u \in V = V_g = V_0 = H^1(0,1)$ such that

$$\int_0^1 u'(x)v'(x)\,dx + \alpha \left(\int_0^1 u(x)\,dx \right) \left(\int_0^1 v(x)\,dx \right) = \int_0^1 f(x)v(x)\,dx =: \langle F, v \rangle, \tag{1.9}$$

for all $v \in V$, has a unique solution for a given $f(\cdot) \in L_2(0,1)$ and for any fixed positive constant α ! Furthermore, prove that if the right-hand side fulfills the solvability condition (1.8) of the Neumann problem, the solution u of (1.9) does not depend on α and solves the variational form (1.4) of the Neumann problem (1.5)–(1.7) with $g_0 = g_1 = 0$.
Hints: Use the Lax-Milgram Theorem and Poincaré's inequality !

$\boxed{05}$ Show that the variational problem: find $u \in V_g = V_0 = \{v \in H^1(0,1) : v(0) = 0\}$ such that

$$\int_0^1 (a(x)u'(x)v'(x) + b(x)u'(x)v(x) + c(x)u(x)v(x))dx = \int_0^1 f(x)v(x)\,dx, \tag{1.10}$$

for all $v \in V_0$, has a unique solution provided that the following assumptions are fulfilled:

1. $a(\cdot) \in L_\infty(0,1)$ and $a(x) \geq a_0 = \text{const} > 0$ for almost all $x \in (0,1)$,
2. $b(x) \equiv b = \text{const} \geq 0$,
3. $c(\cdot) \in L_\infty(0,1)$ and $c(x) \geq 0$ for almost all $x \in (0,1)$,
4. $f(\cdot) \in L_2(0,1)$.

Hint: Use the identity $u'u = \frac{1}{2}(u^2)'$ *and Friedrichs' inequality !*

$\boxed{06^*}$ Due to the Corollary 1.8, the solution of the variational problem (1.10) can be approximated by the fixed point iteration, see formulas (18)–(19) in the lecture notes. Give the classical formulation of this fixed point iteration for the variational problem (1.10) !