3.2. RKM for IVP for 2nd Order ODEs

■ 2nd Order ODE System ($\mathbb{R}^n$) $\Leftrightarrow$ 1st Order ODE System ($\mathbb{R}^{2n}$)

(5) $u''(t) = f(t, u(t), u'(t))$

$u(0) = u_0$

$v(0) = v_0$

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}' = \begin{pmatrix} v(t) \\ f(t, u(t), v(t)) \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

■ RKF $\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$ applied to (6) gives

(7)

$$g_i = u_j + \tau \sum_{k=1}^{\ell} a_{ik} h_k \quad , i = \overline{1,\ell}$$

$$h_i = v_j + \tau \sum_{k=1}^{\ell} a_{ik} f(t_j + c_k \tau, g_k) \quad , i = \overline{1,\ell}$$

$$u_{j+1} = u_j + \tau \sum_{i=1}^{\ell} b_i h_i$$

$$v_{j+1} = v_j + \tau \sum_{i=1}^{\ell} b_i f(t_j + c_i \tau, g_i)$$

$\Updownarrow$

$$g_i = u_j + \tau \sum_{k=1}^{\ell} a_{ik} \left[v_j + \tau \sum_{s=1}^{\ell} a_{ks} f(t_j + c_s \tau, g_s) \right]$$

$$u_{j+1} = u_j + \tau \sum_{i=1}^{\ell} b_i \left[v_j + \tau \sum_{k=1}^{\ell} a_{ik} f(t_j + c_k \tau, g_k) \right]$$

$$v_{j+1} = v_j + \tau \sum_{i=1}^{\ell} b_i f(t_j + c_i \tau, g_i)$$

$\Updownarrow$

$$g_i = u_j + \tau c_i v_j + \tau^2 \sum_{k=1}^{\ell} \bar{a}_{ik} f(t_j + c_k \tau, g_k)$$

$$u_{j+1} = u_j + \tau v_j + \tau^2 \sum_{i=1}^{\ell} \bar{b}_i f(t_j + c_i \tau, g_i)$$

$$v_{j+1} = v_j + \tau \sum_{i=1}^{\ell} b_i f(t_j + c_i \tau, g_i)$$

with $\bar{a}_{ik} = \sum_{s=1}^{\ell} a_{is} a_{sk}$, $\bar{b}_i = \sum_{k=1}^{\ell} b_k a_{ki}$, $\sum_{k=1}^{\ell} a_{ik} = c_i$, $\sum_{i=1}^{\ell} b_i = 1$