The case that J is non-normal: $A = \tau J$

- **Theorem 2.29:**
  Let $(\cdot,\cdot)$ be a scalar product in $\mathbb{C}^n$, $A \in \mathbb{C}^{n\times n}$, and $R(z)$ a rational function, which is bounded on $\mathbb{C}^-$. Assume that
  $$\mathrm{Re}\,(Av,v) \le 0 \quad \forall v \in \mathbb{C}^n.$$
  Then we have the estimate
  $$\|R(A)\| \le \sup_{z\in\mathbb{C}^-} |R(z)|.$$

- **Proof:** see lecture notes, p. 92-93, Th. 2.5 ∎

- **Example 2.30:** cf. also Example 2.28 / 2.20:
  M_h SPD, NOW: $(K_h \underline{v}_h, \underline{v}_h) \ge 0 \ \forall \underline{v}_h \in \mathbb{R}^{N_h}$,
  but non-necessarily symmetric!
  $$J = -M_h^{-1}K_h$$
  $$\mathrm{Re}\,(-M_h^{-1}K_h(\underline{v}_h + i\underline{w}_h), \underline{v}_h + i\underline{w}_h)_{M_h} =$$
  $$= -\mathrm{Re}\,(K_h(\underline{v}_h + i\underline{w}_h), \underline{v}_h + i\underline{w}_h)$$
  $$= -(K_h\underline{v}_h, \underline{v}_h) - (K_h\underline{w}_h, \underline{w}_h) \le 0$$
  Consequently, an A-stable (implicit) RKF for
  $$M_h \underline{u}_h'(t) = \underline{f}_h(t) - K_h \underline{u}_h(t)$$
  is contractive.

For more general ($\curvearrowright$ nonlinear) systems the concept of A-stability is not sufficient: → B-stability

- **Definition 2.31:** B-stability $\Rightarrow$ A-stability

  Examples:
  Impl. Euler
  Impl. MPR
  Gauß-Typ

  A RKF is called B-stable if for all IVP with
  $$(f(t,w) - f(t,v), w-v) \le 0 \quad \forall t, w, v$$
  the approximations satisfy the contraction condition
  $$\|w_{k+1} - v_{k+1}\| \le \|w_k - v_k\| \quad \forall \tau > 0.$$