- **Recall**

expl RKF (26)–(27) $\quad \begin{array}{c|c} c & A = \begin{bmatrix} 0 & & 0 \\ a_{ij} & \ddots & 0 \end{bmatrix} \\ \hline & b^T \end{array} \longmapsto$ (37) $\boxed{u'(t) = \lambda u(t)}$, $u(0) = u_0$

$$\Longrightarrow \quad \begin{aligned} g &= u_j e + \tau\lambda A g \\ u_{j+1} &= u_j + \tau\lambda b^T g \end{aligned} \qquad \begin{aligned} &(26) \\ &(27) \end{aligned} \qquad g = \begin{pmatrix} g_1 \\ \vdots \\ g_s \end{pmatrix}, \; e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^s$$

$g = (I - zA)^{-1} e \cdot u_j$

$\det(I - zA) = 1$

$$\Downarrow$$

(39) $\boxed{u_{j+1} = R(\tau\lambda)\, u_j}$, $\quad j = 0, 1, \ldots, m-1$, u_0 given

where

(40) $R(z) = 1 + z\, b^T (I - zA)^{-1} e$, $\quad z \in \mathbb{C}$

is called <u>stability function</u> of the RKF.

- <u>Example 2.18:</u>
  1. expl. Euler: $R(z) = 1 + z$
  2. improved Euler: $R(z) = 1 + z + \frac{1}{2} z^2$
  3. Cl. RKF 4: $R(z) = 1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3 + \frac{1}{24} z^4$

  cf. $e^z = 1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3 + \frac{1}{24} z^4 + \ldots$

- <u>Remark 2.19:</u>

For the exact solution of the ODE (37), we obtain

$$u(t) = u_0 e^{\lambda t},$$

and, therefore, $u(t_{j+1}) = e^{\lambda\tau} u(t_j)$. Hence,

$$\begin{aligned} u(t_{j+1}) - u_{j+1} &= e^{\lambda\tau} u(t_j) - R(\lambda\tau)\, u_j \\ d_\tau(t_{j+1}) &= \left(e^{\lambda\tau} - R(\lambda\tau)\right) u(t_j) \overset{!}{=} O(\tau^{p+1}) \end{aligned}$$

corresponds to the accuracy of the approximation of $e^z = 1 + z + \frac{1}{2} z^2 + \ldots$ by $R(z)$ in a neighborhood of $z = 0$.