- **Theorem 2.7:** $\exists! u \in H^1 : u' + Au = f,\ u(0) = u_0$

Ass.: 1. Let V and H be separable Hilbertspaces with $V \subset H$, V dense in H, and $\|v\|_H \le c\|v\|_V\ \forall v \in V$;

2. $\exists$ constants $\mu_2 \geq \mu_1 > 0$:

a) $a(v,v) \geq \mu_1 \|v\|_V^2\ \forall v \in V$,

b) $|a(w,v)| \le \mu_2 \|w\|_V \|v\|_V\ \forall v \in V$

St.: Then there exists a unique solution $u \in X = H^1((0,T),V) = H^1((0,T),V;H)$ of the initial value problem (6):

$$\begin{cases} \langle u'(t), v\rangle + a(u(t),v) = \langle F(t), v\rangle & \forall v \in V \\ u(0) = u_0 . \end{cases}$$

Proof: (sketch)

- $\exists$ sequence $(\varphi_i)_{i\in\mathbb{N}} \subset V$: $\overline{\bigcup_{n\in\mathbb{N}} V_n}^{\|\cdot\|_V} = V$, where $V_n := \operatorname{span}(\varphi_1, \dots, \varphi_n)$.
- $u_n \in H^1((0,1),V_n)$: $\langle u_n'(t), v_n\rangle + a(u_n(t), v_n) = \langle F(t), v_n\rangle$

$\exists!$ ☺ P&L $\quad (u_n(0), v_n)_H = (u_0, v_n) \quad \forall v_n \in V_n$

- Because of the uniform a-priori bounds

$$\|u_n\|_X \le c_1,\quad \|Au_n\|_{X^*} \le c_2,\quad \|u_n(t)\|_H \le c_3,$$

it follows by a <u>compactness argument</u> ($\exists$ weakly convergent <u>sub</u>sequences!):

$$\begin{aligned} \langle f, u_n\rangle &\longrightarrow \langle f, u\rangle && \forall f \in X^* \\ \langle Au_n, v\rangle &\longrightarrow \langle w, v\rangle && \forall v \in V \\ (u_n(0), v)_H &\longrightarrow (u_0, v) && \forall v \in H \\ (u_n(T), v)_H &\longrightarrow (z, v) && \forall v \in H \end{aligned}$$

- Now it can easily (uuu) be shown that

$$u' = F - w,\quad u(0) = u_0,\quad u(T) = z$$

and that $\quad Au = w.$ $\qquad$ q.e.d.