ÜBUNGEN ZU

NUMERIK ELLIPTISCHER PROBLEME

für den 14. 6. 2007

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43. Consider the boundary value problem:

$$\begin{aligned} Lu(x) &= f(x) \quad \forall x \in (0,1), \\ u(0) &= u(1) = 0 \end{aligned}$$

with

$$Lu(x) = -u''(x) + bu'(x)$$

and $b \in \mathbb{R}$. Let $\mathcal{T}_h$ denote the equidistant subdivision of the interval $[0,1]$ with mesh size $h = 1/n$, $n \in \mathbb{N}$, the corresponding set of nodes is given by $\overline{\omega}_h = \{x_0, x_1, \ldots, x_n\}$.

Consider the following variational formulation of the boundary value problem: Find $u \in H_0^1(0,1)$ such that

$$a(u,v) = (f,v)_{L^2(0,1)} \quad \forall v \in H_0^1(0,1) \tag{1}$$

with

$$a(u,v) = (u',v')_{L^2(0,1)} + (bu',v)_{L^2(0,1)}.$$

For $b = 0$, it is easy to see that the finite element discretization by the Courant element is equivalent to the finite difference method

$$\begin{aligned} L_h u_h(x) &= f_h(x) \quad \forall x \in \omega_h, \\ u_h(0) &= u_h(1) = 0 \end{aligned}$$

with

$$L_h u_h(x) = -\frac{1}{h^2}\left[u_h(x-h) - 2u_h(x) + u_h(x+h)\right]$$

and

$$f_h(x) = \frac{1}{h}\int_{x-h}^{x+h} f(y)\, p^{(x)}(y)\ dy,$$

where $p^{(x)}(y)$ denotes the nodal basis function associated to the node $x \in \overline{\omega}_h$.

Which finite difference method is equivalent to the finite element discretization by the Courant element for the case $b \neq 0$?

44. Assume the notations of the previous example.

Let δ_h be a given positive real number and consider the following discrete variational problem: Find $u_h \in V_h$ such that

$$a_h(u_h,v_h) = (f,v_h)_{L^2(0,1)} \quad \forall v_h \in V_h \tag{2}$$

with

$$a_h(u_h, v_h) = (u_h', v_h')_{L^2(0,1)} + (bu_h', v_h + \delta_h\, bv_h')_{L^2(0,1)},$$

where $V_h \subset H_0^1(0,1)$ denotes the finite element space given by the Courant element. Determine the parameter δ_h such that the finite element discretization by the Courant element applied to (2) is equivalent to the following finite difference method:

$$\begin{aligned} L_h u_h(x) &= f_h(x) \quad \forall x \in \omega_h, \\ u_h(0) = u_h(1) &= 0 \end{aligned}$$

with

$$\begin{aligned} L_h u_h(x) &= -\frac{1}{h^2}\left[u_h(x-h) - 2u_h(x) + u_h(x+h)\right] \\ &\quad + b \left\{ \begin{array}{ll} \frac{1}{h}\left[u_h(x) - u_h(x-h)\right] & \text{if } b \geq 0 \\ \frac{1}{h}\left[u_h(x+h) - u_h(x)\right] & \text{if } b < 0 \end{array} \right\}. \end{aligned}$$

45. A finite difference method of the form

$$\begin{aligned} L_h u_h(x) &= f_h(x) \quad x \in \omega_h \\ u_h(0) = u_h(1) &= 0 \end{aligned}$$

with

$$L_h u_h(x) = A_h(x) u_h(x) - \sum_{\xi \in S'(x)} B_h(x,\xi) u_h(\xi)$$

is called monotone if and only if

(a) $A_h(x) > 0$ for all $x \in \omega_h$

(b) $B_h(x,\xi) > 0$ for all $\xi \in S'(x)$, $x \in \omega_h$.

(c) $D_h(x) \equiv A(x) - \sum_{\xi \in S'(x)} B(x,\xi) \geq 0$ for all $x \in \omega_h$.

Under what conditions on b and h are the finite difference methods of the two previous examples monotone?

46. Assume the notations of the previous examples.

Show that

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} a_T(u_h, v_h + \delta_h\, bv_h').$$

with

$$a_T(w, v) = (u', v')_{L^2(T)} + (bu', v)_{L^2(T)}.$$

47. Assume the notations of the previous examples.

    Show that
$$a_h(u_h, v_h) = a(u_h, v_h) + \sum_{T \in \mathcal{T}_h} \delta_h (Lu_h, Lv_h)_{L^2(T)}.$$

48. Assume the notations of the previous examples.

    Consider the following variational problem: Find $u_h \in V_h$ such that
$$a_h(u_h, v_h) = \langle F_h, v_h \rangle \quad \forall v_h \in V_h$$
    with
$$a_h(u_h, v_h) = a(u_h, v_h) + \sum_{T \in \mathcal{T}_h} \delta_h (Lu_h, Lv_h)_{L^2(T)}$$
    and
$$\langle F_h, v_h \rangle = (f, v_h)_{L^2(0,1)} + \sum_{T \in \mathcal{T}_h} \delta_h \, (f, Lv_h)_{L^2(T)}.$$
    Let $u \in H_0^1(0,1)$ be the exact solution of (1).

    Show: If $u \in H_0^1(0,1) \cap H^2(0,1)$, then
$$\langle F_h, v_h \rangle - a_h(u, v_h) = 0 \quad \forall v_h \in V_h.$$