

**ÜBUNGEN ZU
NUMERIK ELLIPTISCHER PROBLEME**

für den 31. 5. 2007

37. Let $\delta_i, i = 1, 2$ be two neighboring triangles with a common edge e . s_i denotes the centroid of the triangle $\delta_i, i = 1, 2$. V_e is the quadrilateral which built from the vertices of the edge e and the centroids s_1, s_2 . Moreover, let n_i be the outward normal unit vector of the triangle δ_i on the edge $e, i = 1, 2$.

For a vector-valued function q which is piecewise continuous with respect to the subdivision $\{\delta_1, \delta_2\}$ the following jump is defined on e :

$$[n \cdot q](x) = \lim_{t \rightarrow 0^+} n_1 \cdot q(x - tn_1) + \lim_{t \rightarrow 0^+} n_2 \cdot q(x - tn_2) \quad \forall x \in e.$$

(The symbol \cdot denotes the inner product of vectors.) In particular, the jump

$$\left[\frac{\partial v}{\partial n} \right](x) = [n \cdot \text{grad } v](x) \quad \forall x \in e$$

is well-defined for piecewise continuously differentiable scalar functions v .

For a function f which is piecewise continuous with respect to the subdivision $\{\delta_1, \delta_2\}$ the following average on e is defined:

$$f_e = \frac{1}{|\delta_1| + |\delta_2|} (|\delta_1| f(s_1) + |\delta_2| f(s_2)).$$

Let $V = H^1(\Omega)$ and $V_0 = H_0^1(\Omega)$, let \mathcal{T}_h be an admissible triangulation of the polygonal domain $\Omega \subset \mathbb{R}^2$, and let V_h be the corresponding finite element space defined by the non-conforming Crouzeix-Raviart element (piecewise linear functions given by the values at the midpoints of the edges), and let $V_{0h} \subset V_h$ be the sub-space with values 0 at the midpoint of edges on the boundary $\partial\Omega$.

Let $f \in C(\overline{\Omega})$. We consider the following variational problem: Find $u_h \in V_{0h}$ such that

$$a_h(u_h, v_h) = \langle F_h, v_h \rangle \quad \forall v_h \in V_{0h} \tag{1}$$

with

$$a_h(u_h, v_h) = \sum_{r \in \mathbb{R}_h} \int_{\delta_r} \text{grad } u_h(x) \cdot \text{grad } v_h(x) \, dx, \quad \langle F_h, v_h \rangle = \sum_{r \in \mathbb{R}_h} f(s_r) v_h(s_r) |\delta_r|.$$

Show: $u_h \in V_{0h}$ solves (1) if and only if

$$\frac{1}{d_e} \left[\frac{\partial u_h}{\partial n} \right](x_e) = f_e$$

for all interior edges e of the triangulation \mathcal{T}_h , where x_e denotes the midpoint of the edge e and $d_e = |V_e|/|e|$.

Hint: Let $\phi_e \in V_h$ be the (nodal) basis function defined by

$$\phi_e(x_f) = \delta_{ef}.$$

$u_h \in V_{0h}$ solves (1) if and only if $a_h(u_h, \phi_e) = \langle F_h, \phi_e \rangle$ for all interior edges e . Observe that ϕ_e vanishes outside the two triangles δ_1 and δ_2 with common edge e .

38. Assume the notations and assumptions of example 37.

Show: $u_h \in V_{0h}$ solves (1) if and only if

$$-\int_{\partial V_e} \frac{\partial u_h}{\partial n} ds = f_e |V_e|$$

for all interior edges e of the subdivision \mathcal{T}_h .

Hint: Use

$$\int_{\partial V_e} \frac{\partial u_h}{\partial n} ds = \int_{\partial V_{e,1}} \frac{\partial u_h}{\partial n} ds + \int_{\partial V_{e,2}} \frac{\partial u_h}{\partial n} ds - \int_e ? ds,$$

where $V_{e,i} = V_e \cap \delta_i$, $i = 1, 2$.

39. Assume the notations and assumptions of example 37.

Let $Q_h \subset H(\text{div}, \Omega)$ be the finite element space defined by the (lowest-order) Raviart-Thomas element: A vector-valued function q_h is in Q_h , if and only if

$$q_h(x) = a_r + b_r (x - s_r) \quad \forall x \in \delta_r$$

on each triangle $\delta_r \in \mathcal{T}_h$, where s_r denotes the centroid of δ_r , $a_r \in \mathbb{R}^2$, and $b_r \in \mathbb{R}$, and

$$[n \cdot q_h](x_e) = 0$$

for each interior edge e of the subdivision \mathcal{T}_h .

Show: For each $v_h \in V_{0h}$ (see example 37) we have:

$$\sum_{r \in \mathbb{R}_h} \int_{\delta_r} \text{grad } v_h \cdot q_h dx + \int_{\Omega} v_h \text{div } q_h dx = \sum_{r \in \mathbb{R}_h} \int_{\delta_r} \text{div}(v_h q_h) dx = 0 \quad \forall q_h \in Q_h.$$

Hint: Use Gauss' theorem for $\int_{\delta_r} \text{div}(v_h q_h) dx$ and then show and use for each triangle δ_r that $n \cdot q_h(x) = \text{constant}$ on each edge e of δ_r .

40. Assume the notations and assumptions of the previous examples.

Show: For all triangles $\delta_r \in \mathcal{T}_h$ we have:

$$\int_{\delta_r} (x - s_r) \cdot q_h dx = \frac{1}{2} \int_{\delta_r} |x - s_r|^2 \text{div } q_h dx \quad \forall q_h \in Q_h.$$

Hint: Evaluate directly the left-hand and right-hand side using the special for a function $q_h \in Q_h$ on δ_r .

41. Assume the notations and assumptions of the previous examples.

Let $\bar{V}_h \subset L_2(\Omega)$ be the space of piecewise constant functions with respect to the subdivision \mathcal{T}_h .

Let $u_h \in V_h$ be the solution of (1). The function p_h is defined elementwise by

$$p_h(x) = \text{grad } u_h - \frac{1}{2} f(s_r)(x - s_r) \quad \forall x \in \delta_r.$$

Show: $p_h \in Q_h$.

42. Assume the notations and assumptions of the previous examples.

Let $u_h \in V_h$ be the solution of (1). The piecewise constant function \bar{u}_h is defined elementwise by

$$\bar{u}_h(x) = u_h(s_r) + \frac{f(s_r)}{4|\delta_r|} \int_{\delta_r} |x - s_r|^2 dx \quad \forall x \in \delta_r.$$

Show for $p_h \in Q_h$ und $\bar{u}_h \in \bar{V}_h$:

$$\begin{aligned} \int_{\Omega} p_h \cdot q_h dx + \int_{\Omega} \bar{u}_h \text{div } q_h dx &= 0 & \forall q_h \in Q_h, \\ \int_{\Omega} \bar{v}_h \text{div } p_h dx &= - \sum_{r \in \mathbb{R}_h} f(s_r) \bar{v}_h(s_r) |\delta_r| & \forall \bar{v}_h \in \bar{V}_h. \end{aligned}$$