ÜBUNGEN ZU

NUMERIK ELLIPTISCHER PROBLEME

für den 19. 4. 2007

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7. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_N\}$ be a set of triangles such that the intersection of two triangles from $\mathcal{T}$ is either empty or a common vertex or a common edge. Then $\mathcal{T}$ is called a triangulation. An edge of some triangle from $\mathcal{T}$ is called an interior edge if it is the common edge of two triangles from $\mathcal{T}$. Otherwise, the edge is called a boundary edge. A vertex of some triangle from $\mathcal{T}$ is called a boundary vertex if it is a vertex of a boundary edge. Otherwise, it is called an interior point.

   Assume that the triangulation $\mathcal{T}$ has the following property: T_{k+1} has either exactly one common edge with exactly one triangle from $\mathcal{T}^{(k)} = \{T_1, \ldots, T_k\}$ or T_{k+1} has exactly two common edges with exactly two triangles from $\mathcal{T}^{(k)}$.

   Show that
   $$N = 2\,V_I + V_B - 2,$$
   where V_I denotes the number of interior vertices in $\mathcal{T}$ and V_B denotes the number of boundary vertices in $\mathcal{T}$, and
   $$E = 3\,V_I + 2\,V_B - 3,$$
   where E denotes the number of edges in $\mathcal{T}$, and
   $$N - E + V = 1,$$
   where V is the number of vertices in $\mathcal{T}$.

   Hint: Show the analogous relations for $\mathcal{T}^{(k)}$ by induction.

8. Let $\mathcal{T}_h$ be a triangulation of some domain $\Omega \subset \mathbb{R}^2$. Consider the following finite element spaces.

   (a) The Courant element: $V_h^{(1)}$ is the set of all continuous and piecewise linear functions on Ω. The degrees of freedom are the values at the vertices of $\mathcal{T}_h$.

   (b) The Crouzeix-Raviart element: $V_h^{(2)}$ is the set of all piecewise linear functions which are continuous in all midpoints of the edges of $\mathcal{T}_h$. The degrees of freedom are the values at the midpoints of the edges.

   (c) The non-conforming P_1 element: $V_h^{(3)}$ is the set of all (not necessarily continuous) piecewise linear functions on Ω. The degrees of freedom are the values at the vertices of the triangles of $\mathcal{T}_h$.

   Use the previous example 7 to determine the dimensions of $V_h^{(1)}$, $V_h^{(2)}$ and $V_h^{(3)}$.

9. The Raviart-Thomas element: Let $T \subset \mathbb{R}^2$ be a non-degenerate triangle. Show that an affine linear function $v : \overline{T} \longrightarrow \mathbb{R}^2$ of the form

$$v(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = a + bx = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + b \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with $a \in \mathbb{R}^2$ and $b \in \mathbb{R}$ is uniquely defined by the values $v_n(x) = v(x) \cdot n(x)$ at the midpoints of the three edges, where $n(x)$ denotes the outer normal unit vector at $x \in \partial T$.

10. The Nedelec element: Let $T \subset \mathbb{R}^2$ be a non-degenerate triangle. Show that an affine linear function $v : \overline{T} \longrightarrow \mathbb{R}^2$ of the form

$$v(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + b \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

with $a \in \mathbb{R}^2$ and $b \in \mathbb{R}$ is uniquely defined by the values $v_T(x) = v(x) - v_n(x)n(x)$ at the midpoints of the three edges, where $n(x)$ denotes the outer normal unit vector at $x \in \partial T$.

11. The quadratic element for a triangular subdivision: Let $T \subset \mathbb{R}^2$ be a non-degenerate triangle. Show that a polynomial $v \in P_2$ is uniquely determined by its values at the 3 vertices and the 3 midpoints of the edges.

12. Show that the Argyris element is a C^1-element.

    Hint: Show that the value of a shape function on an edge of a triangle depends only on information on this edge, such as function values and (first and second-order) directional derivatives along the edge.