Local analysis of the CBS constant

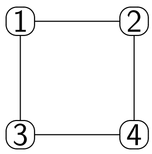
Exercise 19

Consider the variational problem

$$\begin{aligned} a(u,v) &= (f,v) \quad \forall \quad v \in H^1(\Omega), \\ a(u,v) &:= \int_\Omega (\nabla v)^T D \nabla u dx, \\ D &:= \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 < \epsilon \leq 1. \end{aligned}$$

Show that a finite element discretization on a uniform quadrilateral mesh (square elements) using bilinear shape functions, and the local numbering as shown in the Figure below, results in the element stiffness matrix:

$$A_e = \frac{1}{6} \cdot \begin{pmatrix} 2+2\epsilon & 1-2\epsilon & -2+\epsilon & -1-\epsilon \\ 1-2\epsilon & 2+2\epsilon & -1-\epsilon & -2+\epsilon \\ -2+\epsilon & -1-\epsilon & 2+2\epsilon & 1-2\epsilon \\ -1-\epsilon & -2+\epsilon & 1-2\epsilon & 2+2\epsilon \end{pmatrix}. \tag{1}$$



Local numbering of nodes

Exercise 20

Based on the element matrix (1) from Exercise 19, construct the macro-element matrix (for the macro element E composed of four similar elements) with respect to the two-level hierarchical basis.

(a) Compute an upper bound for the local CBS constant γ_E for the case $\epsilon = 1$.

(b) What happens with the bound when $\epsilon \to 0$?

Hint: Use Mathematica or Matlab to apply the technique discussed in the Lecture, that is, solve the related generalized Eigenproblem.