Linear Finite Elements, Hierarchical Basis

Exercise 16

Let e denote an arbitrary triangle with coordinates (x_i, y_i), $i = 1, 2, 3$. Show that transforming the finite element functions between e and the reference triangle $\widetilde{e}$ with coordinates $(0,0), (1,0)$ and $(0,1)$ the element bilinear form $\mathcal{A}_e(.,.)$ becomes:

$$\begin{aligned}
\mathcal{A}_e(u,v) \;&:=\; \int_e \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left[\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right]^T de = \mathcal{A}_{\widetilde{e}}(\widetilde{u}, \widetilde{v}) \\
&:=\; \int_{\widetilde{e}} \left[\frac{\partial \widetilde{u}}{\partial \widetilde{x}}, \frac{\partial \widetilde{u}}{\partial \widetilde{y}}\right] \begin{bmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{bmatrix}^{-1} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\
&\quad \times \begin{bmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{bmatrix}^{-1} \left[\frac{\partial \widetilde{v}}{\partial \widetilde{x}}, \frac{\partial \widetilde{v}}{\partial \widetilde{y}}\right]^T d\widetilde{e},
\end{aligned}$$

where $0 < \widetilde{x}, \widetilde{y} < 1$.

Exercise 17

Show that the element stiffness matrix A_e from Lemma 3.3.1 (using linear shape functions on a triangle) can be represented as

$$A_e = \frac{c}{2} \begin{bmatrix} \beta + 1 & -1 & -\beta \\ -1 & \alpha + 1 & -\alpha \\ -\beta & -\alpha & \alpha + \beta \end{bmatrix},$$

$\alpha = a/c$, $\beta = b/c$, and $(\alpha, \beta) \in D$, where

$$D = \{(\alpha, \beta) \in \mathbb{R}^2 : -\frac{1}{2} < \alpha \leq 1, \max\{-\frac{\alpha}{\alpha + 1}, |\alpha|\} \leq \beta \leq 1\}.$$

Hint: Use Lemma 3.3.2.

Exercise 18

Show that the transformation of the nodal basis stiffness matrix A_h to the hierarchical basis stiffness matrix $\widetilde{A}_h$, using the transformation matrix

$$J^T = \begin{bmatrix} I_1 & 0 \\ J_{21} & I_2 \end{bmatrix},$$

does not change the Schur complement (cf. Equation (3.28) in the Lecture Notes), i.e.,

$$S = A_{22} - A_{21} A_{11}^{-1} A_{12} = \widetilde{A}_{22} - \widetilde{A}_{21} \widetilde{A}_{11}^{-1} \widetilde{A}_{12} = \widetilde{S}.$$