The strengthened Cauchy-Bunyakowski-Schwarz inequality

Exercise 15

As in the Lecture, let V_1, V_2 be finite dimensional spaces, $V = V_1 \times V_2$, associated with the two-by-two block partitioning

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

of a symmetric positive semidefinite (SPSD) matrix A, where A_{11} is assumed to be symmetric positive definite (SPD).

Prove that the following statements related to the CBS inequality

$$|\mathbf{v}_1^T A_{12} \mathbf{v}_2| \le \gamma \, \{\mathbf{v}_1^T A_{11} \mathbf{v}_1 \mathbf{v}_2^T A_{22} \mathbf{v}_2\}^{1/2} \quad \forall \mathbf{v}_i \in V_i$$

hold (cf., Lemma 3.2.2 from the Lecture).

(a) $\gamma \le 1$.

(b) $\gamma = 1$ if there exists $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in \ker(A)$ for which $\mathbf{v}_2 \notin \ker(A_{22})$.

(c) $\gamma < 1$ if for any $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in \ker(A)$ it holds that $\mathbf{v}_2 \in \ker(A_{22})$.

(d) Under the assumption of (c),

$$\gamma = \sup_{\mathbf{v}_i \in V_i \setminus \ker(A_{ii}),\, i=1,2} \frac{{\mathbf{v}_1}^t A_{12} \mathbf{v}_2}{\{{\mathbf{v}_1}^T A_{11} \mathbf{v}_1 \;\; {\mathbf{v}_2}^T A_{22} \mathbf{v}_2\}^{1/2}}.$$