

Condition number, Incomplete Factorization, PCG.

### Exercise 10

Consider the block tridiagonal (sparse) matrix  $A$  of order  $N = n^2$  resulting from discretizing Poisson's equation with the 5-point operator on an  $n$ -by- $n$  uniform mesh assuming (homogeneous) Dirichlet boundary conditions all over the boundary. Using MATLAB this sparse matrix can be generated with the command `A = gallery('poisson',n)`.

(a) Compute the condition number  $\kappa(A)$  of  $A$  for  $n = 2^k$ , where  $k = 1, 2, \dots, k_{\max}$  and  $k_{\max} \in \{6, 7, 8\}$  depending on the available memory on your laptop. Depict  $\kappa(A)$  in a diagram with (doubly) logarithmic scale (log-log-plot) for  $h^{-1} = n = 2^k, k = 1, 2, \dots, k_{\max}$ . How does  $\kappa(A)$  depend on  $h$ ?

(b) Compute an incomplete LU factorization of  $A$ , i.e.,  $A \approx K_0 = L_0 U_0$  as well as a modified incomplete LU factorization of  $A$ , i.e.,  $A \approx K_1 = L_1 U_1$  both with no (additional) fill ((M)ILU(0)), that means that  $L_p, U_p$  for  $p = 0, 1$  have their only nonzero entries in positions where  $A$  has nonzeros. Study the condition number of  $B_0 = K_0^{-1}A$  and  $B_1 = K_1^{-1}A$  for  $h^{-1} = n = 2^k, k = 1, 2, \dots, k_{\max}$  and compare its growth rate with that of  $\kappa(A)$ .

**Hint:** The commands

```
dAKmax=eigs(A,K,1,'lm',options);  
dAKmin=eigs(A,K,1,'sm',options);
```

can be used to compute the largest and smallest eigenvalue of the generalized eigenproblem  $A\mathbf{v} = \lambda K\mathbf{v}$ , respectively. In the options one can specify the precision.

### Exercise 11

(a) Solve the system of linear equations  $A\mathbf{x} = \mathbf{b}$  with the matrix  $A$  from Exercise 10 and a right-hand side vector  $\mathbf{b}$  with random entries, e.g.,  $\mathbf{b} = 100 \cdot \text{rand}(N, 1) / N$ , using the preconditioned conjugate gradient (PCG) method for some fixed tolerance for the relative residual norm, e.g.,  $\text{tol} = 1e-5$ , for  $h^{-1} = n = 2^k, k = 1, 2, \dots, k_{\max}$  and plot the number of iterations in case of

- (i) no preconditioning,
- (ii) using the preconditioner  $K_0$  obtained from ILU(0) factorization,
- (iii) using the preconditioner  $K_1$  obtained from MILU(0) factorization

in a proper diagram.

(b) How is the number of iterations, needed to satisfy the stopping criterion, related to the condition number (estimates) computed in Exercise 10? What does the theory say?