# Fast Solvers: Tutorial — Summer Term 2007

**Conjugate Gradient Method** — **March 26**

Steepest Descent, Conjugate Gradients.

### Exercise 8

Prove the following convergence estimate for the steepest descent method discussed in the Lecture:

$$\|\mathbf{e}^{(n)}\|_A^2 \le \left(1 - \frac{4}{2 + \kappa(A) + 1/\kappa(A)}\right) \|\mathbf{e}^{(n-1)}\|_A^2. \tag{1}$$

Note that (1) provides a sharp upper bound.

**Hint:** From the Lecture we know

$$\|\mathbf{e}^{(n)}\|_A^2 = \left(1 - \frac{\langle \mathbf{r}^{(n-1)}, \mathbf{r}^{(n-1)} \rangle^2}{\langle A^{-1}\mathbf{r}^{(n-1)}, \mathbf{r}^{(n-1)} \rangle \langle \mathbf{r}^{(n-1)}, A\mathbf{r}^{(n-1)} \rangle}\right) \|\mathbf{e}^{(n-1)}\|_A^2.$$

Expand the residual $\mathbf{r}^{(n-1)} = \sum_i c_i \mathbf{\Phi}_i$ into a linear combination of the orthonormal eigenvectors $\{\mathbf{\Phi}_i : 1 \le i \le N\}$ of $A$ and use Lagrange's method to calculate the maximum of

$$\left(1 - \frac{\langle \mathbf{r}^{(n-1)}, \mathbf{r}^{(n-1)} \rangle^2}{\langle A^{-1}\mathbf{r}^{(n-1)}, \mathbf{r}^{(n-1)} \rangle \langle \mathbf{r}^{(n-1)}, A\mathbf{r}^{(n-1)} \rangle}\right),$$

cf. Equation (1.67) in the Lecture.

### Exercise 9

(a) After $n$ iterations of the conjugate gradient method (Alg. 1.4.1) we have

$$\operatorname{span}\{\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \ldots, \mathbf{p}^{(n)}\} = \operatorname{span}\{\mathbf{r}^{(0)}, \mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(n-1)}\}. \tag{2}$$

Verify Equation (2) by mathematical induction.

(b) Sow that

$$\alpha_n = \frac{\langle \mathbf{r}^{(n-1)}, \mathbf{p}^{(n)} \rangle}{\langle \mathbf{p}^{(n)}, \mathbf{p}^{(n)} \rangle_A} = \frac{\langle \mathbf{r}^{(n-1)}, \mathbf{r}^{(n-1)} \rangle}{\langle \mathbf{p}^{(n)}, \mathbf{p}^{(n)} \rangle_A}$$

minimizes $\phi(\mathbf{x}^{(n-1)} + \alpha_n \mathbf{p}^{(n)})$ with respect to $\alpha_n$, where $\phi(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T A \mathbf{x} - \mathbf{x}^T \mathbf{b}$.