Fast Solvers: Tutorial — Summer Term 2007

Basic Iterative Methods — **March 12**

Norms, iteration matrix, spectral radius, convergence, relaxation.

Exercise 1

Let T be a nonsingular matrix. Then

$$\|\mathbf{x}\|_T := \|T\mathbf{x}\|_\infty$$

defines a vector norm. Show that the induced matrix norm

$$\|A\|_T := \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_T}{\|\mathbf{x}\|_T}$$

satisfies $\|A\|_T = \|TAT^{-1}\|_\infty$.

Exercise 2

Prove that for every $\varepsilon > 0$ and any square matrix A there is a nonsingular matrix T such that

$$\|A\|_T \leq \rho(A) + \varepsilon.$$

Hint: There exists a unitary matrix U such that

$$B = UAU^{-1} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ & b_{22} & & b_{2n} \\ & & \ddots & \vdots \\ 0 & & & b_{nn} \end{bmatrix},$$

where $b_{ii} = \lambda_i$ are the eigenvalues of A. The matrix T can then be obtained from U by a proper scaling (of its rows).

Exercise 3

Use Exercise 2 in order to prove the following equivalence:

$$\lim_{k\to\infty} A^k = 0 \quad \Longleftrightarrow \quad \rho(A) < 1.$$

Exercise 4

Prove Theorem 1.1.3 from the Lecture.

Hint: Let $\mathbf{e}^{(k)}$ denote the error after the k-th iteration, and let $\bar{\mathbf{e}}^{(k)}$ be given by $\bar{\mathbf{e}}^{(k)} := A^{\frac{1}{2}}\mathbf{e}^{(k)}$, i.e., $\|\bar{\mathbf{e}}^{(k)}\| = \|\mathbf{e}^{(k)}\|_A$, where $\|\mathbf{x}\|_A := \sqrt{\mathbf{x}^T A\mathbf{x}}$. Then we have

$$A^{\frac{1}{2}} P_k(K^{-1}A)\mathbf{e}^{(0)} = P_k(A^{\frac{1}{2}}K^{-1}A^{\frac{1}{2}})\bar{\mathbf{e}}^{(0)}$$

for any polynomial of degree k (in particular for $P_k(K^{-1}A) = (I - \alpha K^{-1}A)^k$). Moreover, the matrices $A^{\frac{1}{2}}K^{-1}A^{\frac{1}{2}}$ and $K^{-\frac{1}{2}}AK^{-\frac{1}{2}}$ are similar, i.e., they have the same eigenvalues.