

Norms, iteration matrix, spectral radius, convergence, relaxation.

**Exercise 1**

Let  $T$  be a nonsingular matrix. Then

$$\|\mathbf{x}\|_T := \|T\mathbf{x}\|_\infty$$

defines a vector norm. Show that the induced matrix norm

$$\|A\|_T := \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_T}{\|\mathbf{x}\|_T}$$

satisfies  $\|A\|_T = \|TAT^{-1}\|_\infty$ .

**Exercise 2**

Prove that for every  $\varepsilon > 0$  and any square matrix  $A$  there is a nonsingular matrix  $T$  such that

$$\|A\|_T \leq \rho(A) + \varepsilon.$$

**Hint:** There exists a unitary matrix  $U$  such that

$$B = UAU^{-1} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & & b_{2n} \\ & & \ddots & \vdots \\ 0 & & & b_{nn} \end{bmatrix},$$

where  $b_{ii} = \lambda_i$  are the eigenvalues of  $A$ . The matrix  $T$  can then be obtained from  $U$  by a proper scaling (of its rows).

**Exercise 3**

Use Exercise 2 in order to prove the following equivalence:

$$\lim_{k \rightarrow \infty} A^k = 0 \iff \rho(A) < 1.$$

**Exercise 4**

Prove Theorem 1.1.3 from the Lecture.

**Hint:** Let  $\mathbf{e}^{(k)}$  denote the error after the  $k$ -th iteration, and let  $\bar{\mathbf{e}}^{(k)}$  be given by  $\bar{\mathbf{e}}^{(k)} := A^{\frac{1}{2}}\mathbf{e}^{(k)}$ , i.e.,  $\|\bar{\mathbf{e}}^{(k)}\| = \|\mathbf{e}^{(k)}\|_A$ , where  $\|\mathbf{x}\|_A := \sqrt{\mathbf{x}^T A \mathbf{x}}$ . Then we have

$$A^{\frac{1}{2}}P_k(K^{-1}A)\mathbf{e}^{(0)} = P_k(A^{\frac{1}{2}}K^{-1}A^{\frac{1}{2}})\bar{\mathbf{e}}^{(0)}$$

for any polynomial of degree  $k$  (in particular for  $P_k(K^{-1}A) = (I - \alpha K^{-1}A)^k$ ). Moreover, the matrices  $A^{\frac{1}{2}}K^{-1}A^{\frac{1}{2}}$  and  $K^{-\frac{1}{2}}AK^{-\frac{1}{2}}$  are similar, i.e., they have the same eigenvalues.